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Combinatorial constructions of modules for infinite-dimensional Lie algebras, I. Principal subspace

Galin Georgiev *

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

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Abstract

This is the first of a series of papers studying combinatorial (with no "subtractions") bases and characters of standard modules for affine Lie algebras, as well as various subspaces and "coset spaces" of these modules.

In part I we consider certain standard modules for the affine Lie algebra $\hat{\mathfrak{g}}, \mathfrak{g} := sl(n+1, \mathbb{C})$, $n \geq 1$, at any positive integral level k and construct bases for their principal subspaces (introduced and studied recently by Feigin and Stoyanovsky (1994)). The bases are given in terms of partitions: a color i, $1 \leq i \leq n$, and a charge s, $1 \leq s \leq k$, are assigned to each part of a partition, so that the parts of the same color and charge comply with certain difference conditions. The parts represent "Fourier coefficients" of vertex operators and can be interpreted as "quasi-particles" enjoying (two-particle) statistical interaction related to the Cartan matrix of $\mathfrak{g}$. In the particular case of vacuum modules, the character formula associated with our basis is the one announced in Feigin and Stoyanovsky (1994). New combinatorial characters are proposed for the whole standard vacuum $\hat{\mathfrak{g}}$-modules at level one.

0. Introduction

0.1

This paper was meant to be a higher-rank generalization of the seminal work of Lepowsky and Primc [39] where they built vertex operator combinatorial bases for what came to be called $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ *coset subspaces* of standard (integrable highest weight) $\hat{\mathfrak{g}}$-modules, $\mathfrak{h}$ being the Cartan subalgebra of $\mathfrak{g} = sl(2, \mathbb{C})$. The paper [39] was part of the $\mathscr{Z}$-algebra program originated in [38–42] for studying affine algebras through vertex operators centralizing their Heisenberg subalgebras. Recall that the structure of

* E-mail: georgiev@math.rutgers.edu.

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the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspaces is essentially encoded in their quotient spaces of coinvariants with respect to the action of a certain abelian group. The latter are known to the physicists as *parafermionic spaces* because they turned out to be the underlying spaces of the so-called parafermionic conformal field theories of Zamolodchikov and Fateev [58] (with Virasoro algebra central charge $c = (2k - 2/k + 2)$, $k \in \mathbb{Z}$, $k > 1$; we note that the original twisted vertex operator construction [40] found a conformal-field-theoretical understanding in [59]). The original $\mathscr{Z}$-algebra program was later generalized and elaborated both by mathematicians [10, 30, 43, 46–51, 53, 54], and physicists – cf. [24], where the string functions and parafermionic field theories associated with more general affine Lie algebras were studied. Variations of the celebrated Rogers–Ramanujan identities and their generalizations – the Gordon–Andrews–Bressoud identities [28, 1] – were first interpreted in representation-theoretical terms by Lepowsky and Wilson [40–42] and were ubiquitous in the subsequent works. Similar in spirit construction – in terms of the "difference two" condition, discovered in representation-theoretical context in [40] – was later proposed for the Virasoro algebra minimal models $\mathscr{M}(2, 2n+1)$, $n > 1$, [15, 14]. Despite all the progress, we were still far away from a simple and conceptual vertex operator (combinatorial) construction of the higher-level standard modules for higher-rank affine Lie algebras. A good measure of this deficiency was the lack of nice and general enough combinatorial character formulas.

In the present paper and its follow-ups we shall venture to fill in part of this gap. Working in the setting of Vertex Operator Algebra Theory [22], we follow the tradition and build vertex operator combinatorial bases, i.e., each basis vector is a finite product of vertex operators acting on a highest weight vector (the highest weights in consideration here are specified at the beginning of Section 5). Although we work with $\mathfrak{g} = sl(n+1, \mathbb{C})$, our construction has a straightforward generalization for any finite-dimensional simple Lie algebra $\mathfrak{g}$ of type A–D–E. Major technical tool is the homogeneous vertex operator construction of level one standard modules [21, 55]. The most delicate part of the proofs – the independence arguments – employ the intertwining vertex operators (in the sense of [20]) constructed in [13], which interchange different modules (but a general idea of Lepowsky's, to build bases from the intertwining vertex operators themselves, is yet to become a reality; for the level one case, cf. Proposition 0.2 and for the rank one case, cf. [7–9]).

We would like to point out that constructing a basis for $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspaces (or, equivalently, for the parafermionic spaces) disentangles also the structure of a variety of other important representations. For example, the standard module itself is well known [40] to be a tensor product of its $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspace and a bosonic Fock space representation of a Heisenberg algebra associated with $\hat{\mathfrak{h}}$. More original application (to be discussed in [26]) of structural results for parafermionic spaces is through "nested" coset subspaces of type $\left(\hat{\mathfrak{g}}^{(1)} \supset \hat{\mathfrak{h}}^{(1)}\right) \supset \left(\hat{\mathfrak{g}}^{(2)} \supset \hat{\mathfrak{h}}^{(2)}\right)$ for some pair $\mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)}$. If one picks for example $\mathfrak{g}^{(2)} := sl(n, \mathbb{C})$ which is naturally embedded in $\mathfrak{g}^{(1)} := sl(n+1, \mathbb{C})$, one obtains the Virasoro algebra unitary modules [23] with central charge $c = 1 - 6/[(n+2)(n+3)]$, $n \in \mathbb{Z}$, $n > 1$ (the underlying spaces of the minimal conformal field theories of Belavin, Polyakov and Zamolodchikov [2]; as Alexander Zamolodchikov pointed to us, a similar in spirit coset construction had been largely used by early conformal field theorists prior to the ground-breaking work [27])

For reasons not fully understood, the character formula associated with the Lepowsky–Primc basis was vastly generalized in different directions on the physics side through a sweeping series of fascinating conjectures (inspired mainly by Thermodynamic Bethe Ansatz techniques): cf. [56, 52, 37, 34, 32, 33, 11, 12, 35, 57, 4] to name some few. The announced proofs [3, 36, 18, 4, 19] of some of these conjectures, do not even attempt to reveal the underlying vertex operator combinatorial bases (very inspiring exception is the spinon construction of basic $\widehat{sl}(2,\mathbb{C})$-modules related to the so-called Haldane–Shastry spin chain [5, 7–9]; see Proposition 0.2 at the end of this Introduction where we state the higher-rank generalization of the spinon character formula for level one). Not surprisingly, the true precursor of all these new characters seems to be another remarkable subspace (of the standard modules for affine Lie algebras), introduced in a recent announcement of Feigin and Stoyanovsky [16]: The so-called *principal subspace*, whose dual space has been beautifully described in [16] in terms of symmetric polynomial forms vanishing on certain hyperplanes. These subspaces are generated by the affinization of the nilpotent subalgebra of $\mathfrak{g}$ consisting of (strictly) upper-triangular matrices when $\mathfrak{g} = sl(n+1,\mathbb{C})$ and have obvious generalizations for any simple finite-dimensional Lie algebra $\mathfrak{g}$. It turned out to be more conceptual and technically easier to work with vertex operators (the products of whose coefficients generate our bases) in the setting of principal subspaces. More importantly for our initial commitments, establishing a vertex operator basis for a principal subspaces implies the existence of a similar basis for the corresponding parafermionic space (this issue will be addressed in part II [25]; Feigin and Stoyanovsky themselves were well aware of this close relationship between their considerations and the approach of Lepowsky et al. [38–42]). Furthermore, representing a given standard module as a direct limit of "twisted" (hit by special inner automorphisms) principal subspaces, one can hope to obtain a combinatorial basis for the whole standard module in terms of "semiinfinite monomials" (see [16] for the case $\mathfrak{g} = sl(2,\mathbb{C})$; this approach is similar in spirit to [39] and, not surprisingly, the corresponding character formulas are the same). Actually, there is more natural, in our opinion, way to build the whole standard module in terms of *finite* monomials, starting from the principal subspace: One simply has to throw in the game the negative simple roots and keep playing by the same rules. The corresponding character formula for the level one case is given in Proposition 0.1 at the end of this Introduction. The higher levels are treated in [25].

0.2

We proceed with a demonstration of our bases in the two simplest examples beyond $\widehat{sl}(2,\mathbb{C})$: the principal subspaces of the vacuum standard $\widehat{sl}(3,\mathbb{C})$-modules at levels one and two. They carry to a large extent the spirit of the general case and will ease the impetuous reader.

Fix a triangular decomposition $sl(3,\mathbb{C}) =: \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and a corresponding Chevalley basis of $\mathfrak{g}$:

$$\{x_{-\alpha_1}, x_{-\alpha_2}, x_{-\alpha_1-\alpha_2}\} \bigsqcup \{h_{\alpha_1}, h_{\alpha_2}\} \bigsqcup \{x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_1+\alpha_2}\},$$

where α_1, α_2 are the simple (positive) roots. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t] \oplus \mathbb{C}c$ be the corresponding (untwisted) affine Lie algebra (cf. [31]) and set $g(m) := g \otimes t^m$, $D := -td/dt$, $g \in \mathfrak{g}$, $m \in \mathbb{Z}$. Let $v(\hat{\Lambda}_0)$ be the highest weight vector of the vacuum standard $\hat{\mathfrak{g}}$-module $L(\hat{\Lambda}_0)$ at level one (the eigenvalue of the central charge c is called level). Motivated by [11] and [16], we shall refer to $x_{\alpha_i}(m)$, $i = 1, 2$, as *quasi-particle of charge* 1, *color* i *and energy* $-m$. The vacuum principal subspace at level one $W(\hat{\Lambda}_0) := U(\mathfrak{n}_+ \otimes \mathbb{C}[t^{-1}, t]) \cdot v(\hat{\Lambda}_0)$ (where $U(\cdot)$ denotes universal enveloping algebra), has a basis generated by the following color ordered quasi-particle monomials acting on $v(\hat{\Lambda}_0)$:

$$\mathfrak{B}_{W(\hat{\Lambda}_0)} = \bigsqcup_{r_2, r_1 \geq 0} \left\{ x_{\alpha_2}(m_{r_2,2}) \cdots x_{\alpha_2}(m_{1,2}) x_{\alpha_1}(m_{r_1,1}) \cdots x_{\alpha_1}(m_{1,1}) \right.$$

$$\left| \begin{array}{l} m_{p,i} \in r_{i-1} - 1 - \mathbb{N} \text{ for } 1 \leq p \leq r_i; \\ m_{p+1,i} \leq m_{p,i} - 2 \text{ for } 1 \leq p < r_i;\ i = 1, 2 \end{array} \right\}, \tag{0.1}$$

where $r_0 := 0$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. For an explicit list of some basis elements with low energies, see Example 4.1, Section 4 and the corresponding Table 1 in the appendix. The quasi-particle monomial basis for $\mathfrak{g} = sl(n+1, \mathbb{C})$ and generic fundamental (i.e. level one) highest weight $\hat{\Lambda}_j$, $0 \leq j \leq n$, is given in Definition 4.1 and formula (4.3).

In physicists' terms, the above basis can be described in a very simple and natural way, reminiscent of the "functional" description of the restricted dual of the principal space in [16]: Consider the Fock space of two different (of color 1 or 2) free bosonic quasi-particles with a single quasi-particle energy spectrum consisting of all the integers greater than or equal to the charge ($= 1$) of the quasi-particle. Take a hamiltonian $\mathscr{H}$ which is simply a sum of a single-particle term $\mathscr{H}_1$ and a two-particle interaction term $\mathscr{H}_2$ (this interaction should be thought of as a statistical interaction in the sense of Haldane [29]). The single-particle energy is of course the sum of the single-particle energies of all the quasi-particles (in a given state) and the two-particle energy is a sum of the interaction energies of all the pairs of quasi-particles (in a given state). The energy of interaction between a quasi-particle of color l and another quasi-particle of color m is A_{lm}, where $(A_{lm}) := \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ is the Cartan matrix of $\mathfrak{g}$ (the inner products of simple roots are inherently encoded in both the spanning and independence part of our proof; cf. Section 4). Since the character of the Fock space of noninteracting quasi-particles is

$$\operatorname{Tr} q^{\mathscr{H}_1}\Big|_{\text{Fock}} = \frac{1}{(q)_\infty^2} = \sum_{r_1 \geq 0} \frac{q^{r_1}}{(q)_{r_1}} \sum_{r_2 \geq 0} \frac{q^{r_2}}{(q)_{r_2}}$$

$$= \sum_{r_1, r_2 \geq 0} \frac{q^{r_1+r_2}}{(q)_{r_1}(q)_{r_2}}, \tag{0.2}$$

where $(q)_r := (1-q)(1-q^2)\cdots(1-q^r)$, $(q)_0 := 1$, the full q-character of $W(\hat{\Lambda}_0)$ will be a "deformation" of this expression with an interaction term (the binomial coefficient $\binom{r_i}{2}$ counting all the pairs of color i) :

$$\mathrm{Tr}\, q^D\Big|_{W(\hat{\Lambda}_0)} = \mathrm{Tr}\, q^{\mathscr{H}_1+\mathscr{H}_2}\Big|_{\mathrm{Fock}} = \sum_{r_1,r_2\geq 0} \frac{q^{r_1+r_2+2\binom{r_1}{2}+2\binom{r_2}{2}-r_1r_2}}{(q)_{r_1}(q)_{r_2}} \tag{0.3}$$

$$= \sum_{r_1,r_2\geq 0} \frac{q^{r_1^2+r_2^2-r_1r_2}}{(q)_{r_1}(q)_{r_2}} = \sum_{r_1,r_2\geq 0} \frac{q^{1/2\sum_{l,m=1}^{2} A_{lm}r_lr_m}}{(q)_{r_1}(q)_{r_2}}.$$

The last expression is the Feigin–Stoyanovsky formula [16] (for generic rank and fundamental highest weight, see (4.19) here). At the end of the Introduction we state the analog of this formula for the *whole* vacuum basic module $L(\hat{\Lambda}_0)$.

We proceed with the basis for the level two vacuum principal subspace ($\mathfrak{g} = sl(3,\mathbb{C})$). Here comes the surprise: rather unexpectedly, we shall give up the traditional approach to build module basis using a basis of the Lie algebra itself (i.e., quasi-particles of charge one). Instead, the building blocks of our basis will be certain – very natural from the point of view of Conformal Field Theory (and for that matter, Vertex Operator Algebra Theory [22]) – infinite linear combination of charge-one-quasi-particle monomials [39, 58]. The use of these higher-charge (see below) vertex operators changes dramatically the structure of the basis: even in the special case $\mathfrak{g} = sl(2,\mathbb{C})$, we end up with construction very different from [39] for example (cf. [25] for more detailed discussion). We believe that only this broader perspective can save the elegance and simplicity of the picture in the presence of more than one color (i.e., rank $\mathfrak{g} > 1$) at higher levels. The linear combinations in question are actually truncated (i.e., finite) when acting on highest weight modules and can be thought of as *quasi-particles of charge* > 1. For example, a quasi-particle of charge 2, color i and energy $-m$ is defined as follows:

$$x_{2\alpha_i}(m) := \sum_{\substack{m_2,m_1\in\mathbb{Z}\\ m_2+m_1=m}} x_{\alpha_i}(m_2)x_{\alpha_i}(m_1) \tag{0.4}$$

(for a general definition of quasi-particles of arbitrary charge, see Section 3). Note that a quasi-particle of charge 2 "confines" two quasi-particles of charge 1 in such a way that only the *total* energy can be read off (the individual energies are not "measurable"). In terms of the vertex operator (bosonic current)

$$X_{\alpha_i}(z) := Y(x_{\alpha_i}(-1)\cdot v(2\hat{\Lambda}_0), z) = \sum_{m\in\mathbb{Z}} x_{\alpha_i}(m)z^{-m-1} \tag{0.5}$$

(z is simply a formal variable and $v(2\hat{\Lambda}_0)$ is the highest weight vector of the level two vacuum standard module $L(2\hat{\Lambda}_0)$), one has

$$X_{2\alpha_i}(z) := X_{\alpha_i}(z)X_{\alpha_i}(z) = Y(x_{\alpha_i}(-1)^2 \cdot v(2\hat{\Lambda}_0), z) = \sum_{m\in\mathbb{Z}} x_{2\alpha_i}(m)z^{-m-2}. \tag{0.6}$$

This is a standard recipe in Conformal Field Theory for obtaining currents of higher charge (the usual "normal ordering" of the product $X_{\alpha_i}(z)X_{\alpha_i}(z)$ is not needed here because of the commutativity of the two factors).

Let us now come back to the vacuum principal subspace at level two $W(2\hat{\Lambda}_0) := U(\mathfrak{n}_+ \otimes \mathbb{C}[t^{-1}, t]) \cdot v(2\hat{\Lambda}_0)$. Our basis will be generated by (charge and color ordered) quasi-particle monomials acting on $v(2\hat{\Lambda}_0)$, with $p_i^{(2)} := r_i^{(2)}$ quasi-particles of charge 2 and color i, $i = 1, 2$, and $p_i^{(1)} := r_i^{(1)} - r_i^{(2)}$ quasi-particles of charge 1 and color i, for some $r_i^{(1)} \geq r_i^{(2)} \geq 0$ (i.e., a total charge of $p_i^{(1)} + 2p_i^{(2)} = r_i^{(1)} + r_i^{(2)}$). The conditions they must satisfy are given below (the top two lines on the right-hand side of the delimiter | concern the quasi-particles of charge 2; the bottom two lines concern the quasi-particles of charge 1):

$$\mathfrak{B}_{W(2\hat{\Lambda}_0)} = \bigsqcup_{\substack{r_2^{(1)}\geq r_2^{(2)}\geq 0\\ r_1^{(1)}\geq r_1^{(2)}\geq 0}} \tag{0.7}$$

$$\left\{ x_{\alpha_2}(m_{r_2^{(1)},2}) \cdots x_{\alpha_2}(m_{r_2^{(2)}+1,2}) x_{2\alpha_2}(m_{r_2^{(2)},2}) \cdots x_{2\alpha_2}(m_{1,2}) \cdot \right.$$

$$\times x_{\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{\alpha_1}(m_{r_1^{(2)}+1,1}) x_{2\alpha_1}(m_{r_1^{(2)},1}) \cdots x_{2\alpha_1}(m_{1,1})$$

$$\left| \begin{array}{l} m_{p,i} \in r_{i-1}^{(1)} + r_{i-1}^{(2)} - 2 - \mathbb{N} \ \text{ for } \ 1 \leq p \leq r_i^{(2)};\\ m_{p+1,i} \leq m_{p,i} - 4 \ \text{ for } \ 1 \leq p < r_i^{(2)}; \ i = 1, 2;\\ \\ m_{p,i} \in r_{i-1}^{(1)} - 2r_i^{(2)} - 1 - \mathbb{N} \ \text{ for } \ r_i^{(2)} < p \leq r_i^{(1)};\\ m_{p+1,i} \leq m_{p,i} - 2 \ \text{ for } \ r_i^{(2)} < p < r_i^{(1)}; \ i = 1, 2 \end{array} \right\},$$

where $r_0^{(1)} = r_0^{(2)} := 0$. For an explicit list of some basis elements with low enegies, see Example 5.1, Section 5 and the corresponding Table 2 in the Appendix. (The quasi-particle monomial basis for $\mathfrak{g} = \mathfrak{sl}(n+, \mathbb{C})$ and level k highest weight $\hat{\Lambda} = k_0\hat{\Lambda}_0 + k_j\hat{\Lambda}_j$, $k_0 + k_j = k$, $1 \leq j \leq n$, is given in Definition 5.1 and formula (5.14).)

Although the initial conditions above may look strange at first sight, they can be read off in a remarkably simple way (a straightforward generalization of the level one picture): Consider the Fock space of four different (of color 1 or 2 and charge 1 or 2) free bosonic quasi-particles with single quasi-particle energy spectrum consisting of all the integers greater or equal to the charge of the quasi-particle. The hamiltonian consists again only of a single-particle term $\mathscr{H}_1$ and a two-particle interaction term $\mathscr{H}_2$. The single-particle energy is a sum of the single-particle energies of all the quasi-particles (in a given state) and the two-particle energy is a sum of the interaction energies of all the pairs of quasi-particles (in a given state). The energy of interaction between a quasi-particle of charge s and color l and another quasi-particle of charge t and color m is $A_{lm}\min\{s,t\}$. Similarly to the level one case, the corresponding q-character is

$$\operatorname{Tr} q^{D}\Big|_{W(2\hat{\Lambda}_0)} = \operatorname{Tr} q^{\mathscr{H}_1+\mathscr{H}_2}\Big|_{\text{Fock}} = \sum_{\substack{p_1^{(1)},\, p_1^{(2)}\geq 0\\ p_2^{(1)},\, p_2^{(2)}\geq 0}} \frac{q^{\frac{1}{2}\sum_{l,\,m=1,2}^{s,\,t=1,2} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{(q)_{p_1^{(1)}}(q)_{p_1^{(2)}}(q)_{p_2^{(1)}}(q)_{p_2^{(2)}}}, \tag{0.8}$$

where $B^{st} := \min\{s,t\}$, $1 \leq s,t \leq 2$ (cf. formula (5.27) for the general case). This is the character announced by Feigin and Stoyanovsky [16].

0.3

A few words should be said about the continuation of Part I and its easily conceivable generalizations, as well as some related open problems.

As already mentioned, Part II [25] employs the given here bases and constructs similar quasi-particle monomial bases for the parafermionic spaces in standard modules ($\mathfrak{g} = sl(n+1,\mathbb{C})$ and highest weights like the ones considered here). In the particular case of the vacuum module, the associated character formula is the $\widehat{sl}(n+1,\mathbb{C})$-case of Kuniba–Nakanishi–Suzuki conjecture [37] (for the vacuum module, an independent proof using dilogarithms was recently announced in [36]).

In [26] we shall "factorize" these characters in order to obtain combinatorial characters for the nested coset $\left(\hat{\mathfrak{g}}^{(1)} \supset \hat{\mathfrak{h}}^{(1)}\right) \supset \left(\hat{\mathfrak{g}}^{(2)} \supset \hat{\mathfrak{h}}^{(2)}\right)$ subspace of two $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspaces of level two standard modules, $sl(n+1,\mathbb{C}) = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} = sl(n,\mathbb{C})$. With the natural structure of Virasoro algebra modules, these nested coset spaces are exactly the unitary modules of central charge $c = 1 - 6/[(n+2)(n+3)]$, $n \in \mathbb{Z}$, $n > 1$, (in other words, we use a coset realization different from the classical [27]). The obtained combinatorial characters are the "fermionic" characters conjectured by Kedem et al. [33] (cf. also [44]; for a large subclass of the modules in question, these character formulas were proven by very different methods in [3]; general minimal models are treated in [4]). In order to cover all the Virasoro algebra modules from this series (not only the vacuum ones!), one needs to go beyond the *vacuum* module characters of Feigin–Stoyanovsky and Kuniba–Nakanishi–Suzuki and consider more general dominant integral highest weights.

Coming back to the considerations in Part I, we would like to point out that they have a straighforward generalization for the untwisted affinization $\hat{\mathfrak{g}}$ of any finite-dimensional simple Lie algebra $\mathfrak{g}$ of type A–D–E and highest weights of the type considered here. What is not clear at this point is how to generalize elegantly this construction for *any* dominant integral highest weight? Another exciting open problem is to construct a basis of colored semi-infinite quasi-particle monomials for the whole standard module, thus generalizing the $\widehat{sl}(2,\mathbb{C})$-considerations of Feigin and Stoyanovsky [16] (the generated in this way characters are the same as the ones obtained from the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset decomposition of the standard module and the the Kuniba–Nakanishi–Suzuki characters of the parafermionic spaces).

We conclude this Introduction with the presentation of two new combinatorial character formulas for the vacuum basic $\hat{\mathfrak{g}}$-module $L(\hat{\Lambda}_0)$ (in order to be coherent with the rest of the paper, we shall state the results for $\mathfrak{g} = sl(n+1,\mathbb{C})$, but the generalization for any A–D–E type algebra is obvious). The first formula reflects a basis which is the most natural extension of the constructed here basis for the principal subspace $W(\hat{\Lambda}_0) \subset L(\hat{\Lambda}_0)$: One simply has to imitate the principal subspace construction, adding vertex operators corresponding to the negative simple roots and taking into account the new constraint (cf. [39])

$$(z_2 - z_1)^{\langle \alpha_i, \alpha_i \rangle} X_{-\alpha_i}(z_2) X_{\alpha_i}(z_1) \Big|_{z_1 = z_2} = \text{const.}$$

for every simple root α_i, $1 \le i \le n$.

Proposition 0.1. *One has the following q-character for the vacuum basic (standard, level one) $\hat{\mathfrak{g}}$-module:*

$$\operatorname{Tr} q^D \Big|_{L(\hat{\Lambda}_0)} = \sum_{r_{\pm 1}, \ldots, r_{\pm n} \ge 0} \frac{q^{1/2 \sum_{l,m=1}^n A_{lm}(r_{+l} - r_{-l})(r_{+m} - r_{-m}) + \sum_{l=1}^n r_{+l} r_{-l}}}{\prod_{l=1}^n (q)_{r_{+l}} (q)_{r_{-l}}}, \tag{0.9}$$

where $(q)_r := (1-q)(1-q^2)\cdots(1-q^r)$, $(q)_0 := 1$ and (A_{lm}) is the Cartan matrix of $\mathfrak{g}$.

Note that in the particular case $\hat{\mathfrak{g}} = \widehat{sl}(2,\mathbb{C})$, one can redefine $r_1 := r_{+1}$, $r_2 := r_{-1}$, and thus obtain the character (0.3)

$$\sum_{r_1, r_2 \ge 0} \frac{q^{r_1^2 + r_2^2 - r_1 r_2}}{(q)_{r_1} (q)_{r_2}}$$

of the principal subspace of the vacuum basic $\widehat{sl}(3,\mathbb{C})$-module. This coincidence was observed and explained by Feigin and Stoyanovsky [16] (the expression (0.3), representing the character of $L(\hat{\Lambda}_0)$ for $\hat{\mathfrak{g}} = \widehat{sl}(2,\mathbb{C})$, appeared also in [45] as a limit of certain finite "fermionic" sums). The proof of the general formula in the above proposition is analogous to the proof of the special case [16]: Using the "Durfee rectangle" combinatorial identity [1]

$$\frac{1}{(q)_\infty} := \prod_{l \ge 0} (1 - q^l)^{-1} = \sum_{\substack{a,b \ge 0 \\ a-b = \text{const}}} \frac{q^{ab}}{(q)_a (q)_b}, \tag{0.10}$$

one immediately checks that (0.9) equals the well-known character expression due to Feingold–Lepowsky [17] (which inspired the homogeneous vertex operator construction [21, 55])

$$\operatorname{Tr} q^{D}\Big|_{L(\hat{\Lambda}_0)} = \frac{1}{(q)_\infty^n} \sum_{\beta \in Q} q^{1/2\langle \beta,\beta\rangle}, \tag{0.11}$$

where Q is the root lattice of $\mathfrak{g}$.

The higher level generalization of Proposition 0.1 will be presented in [25] because it requires the respective generalization of formula (0.11).

Our second character formula mirrors another basis for the same module, this time built up from intertwining vertex operators (in the sense of [20]) corresponding to the fundamental weights and their negatives.

Proposition 0.2. *Another expression for the q-character of the vacuum basic $\hat{\mathfrak{g}}$-module is*

$$\operatorname{Tr} q^{D}\Big|_{L(\hat{\Lambda}_0)} = \sum_{\substack{r_{\pm 1},\dots,r_{\pm n} \geq 0 \\ \sum_{m=1}^{n} A_{lm}^{(-1)}(r_{+m}-r_{-m}) \in \mathbb{Z}\ \forall\, l}} \frac{q^{1/2\sum_{l,m=1}^{n} A_{lm}^{(-1)}(r_{+l}-r_{-l})(r_{+m}-r_{-m}) + \sum_{l=1}^{n} r_{+l}r_{-l}}}{\prod_{l=1}^{n}(q)_{r_{+l}}(q)_{r_{-l}}}, \tag{0.12}$$

where $(A_{lm}^{(-1)})$ is the inverse of the Cartan matrix of $\mathfrak{g}$.

In the particular case $\hat{\mathfrak{g}} = \widehat{sl}(2,\mathbb{C})$, one can again substitute $r_1 := r_{+1}$, $r_2 := r_{-1}$, and thus obtain the Kedem–McCoy–Melzer formula [35, 45]

$$\sum_{\substack{r_1, r_2 \geq 0 \\ r_1 - r_2 \text{ even}}} \frac{q^{\frac{1}{4}(r_1+r_2)^2}}{(q)_{r_1}(q)_{r_2}},$$

which was shown to correspond to a spinon basis in [5] (cf. also [7–9]). Note that its higher-rank generalization (0.12) follows again from (0.10) and (0.11) (this time, the elements of the root lattice are expressed in terms of fundamental weights).

The higher-level generalization of Proposition 0.2 is yet unknown (one possible approach in the particular case $\mathfrak{g} = sl(2,\mathbb{C})$ was proposed in [8]).

The bases underlying the above expressions will be discussed in detail elsewhere.

0.4

The paper is organized as follows. In Section 1 we introduce most of our notations and definitions. In Section 2 we recall the homogeneous vertex operator construction of the basic (i.e., level one standard) modules. Section 3 introduces the concept of quasi-particle of any integral positive charge. In Section 4 we build quasi-particle bases for the principal subspaces of the basic modules and supply the accompanying character formulas. Section 5 generalizes this construction and provides quasi-particle bases and corresponding characters for the principal subspaces at any positive integral level k. The appendix contains two tables which illustrate the examples discussed in this Introduction: Example 4.1 (Section 4) and Example 5.1 (Section 5).

1. Preliminaries

We shall use the notation $\mathbb{Z}_+$ for the set of positive integers and $\mathbb{N}$ for $\mathbb{Z}_+ \cup \{0\}$.

Fix $n \in \mathbb{Z}_+$ and let $\mathfrak{g} := sl(n+1, \mathbb{C})$. Choose a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Denote by $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ the set of simple (positive) roots, with the indices reflecting their locations on the Dynkin diagram. The notation Δ_+ (respectively, Δ_-) signifies the set of positive (resp., negative) roots; $\Delta := \Delta_+ \cup \Delta_-$. The highest (positive) root will be denoted by θ and let us normalize the nonsingular invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$ so that $\langle \theta, \theta \rangle = 2$ (a condition true for any root); then we have a corresponding form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$. Let ρ be half the sum of the positive roots and $h = h^\vee = n+1$ the (dual) Coxeter number. For any $\mu \in \mathfrak{h}^*$ denote by $h_\mu \in \mathfrak{h}$ its "dual": $\lambda(h_\mu) = \langle \lambda, \mu \rangle$ for every $\lambda \in \mathfrak{h}^*$.

We fix for concreteness a Chevalley basis $\{x_\alpha\}_{\alpha \in \Delta} \cup \{h_{\alpha_i}\}_{i=1}^n$ of $\mathfrak{g}$. Let $Q := \sum_{i=1}^n \mathbb{Z}\alpha_i$, $P := \sum_{i=1}^n \mathbb{Z}\Lambda_i$ be the root and weight lattice respectively, where Λ_i, $i = 1, \ldots, n$, are the fundamental weights: $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}, i, j = 1, \ldots, n$. Set $Q_+ \subset Q$ (respectively, $Q_-) \subset Q$ to be the semigroup (without 0) generated by the simple roots Π (respectively, by $-\Pi$). We denote the group algebras corresponding to Q and P by $\mathbb{C}[Q] := \operatorname{span}_{\mathbb{C}}\{e^\beta \mid \beta \in Q\}$ and $\mathbb{C}[P] := \operatorname{span}_{\mathbb{C}}\{e^\lambda \mid \lambda \in P\}$. There exists a central extension

$$1 \longrightarrow \langle e^{\pi i/(n+1)^2} \rangle \longrightarrow \hat{P} \longrightarrow P \longrightarrow 1 \tag{1.1}$$

(which after restriction provides a central extension $\hat{Q}$ of Q), by the finite cyclic group $\langle e^{\pi i/(n+1)^2} \rangle$ of order $2(n+1)^2$, satisfying the following condition: if one chooses a 2-cocycle

$$\varepsilon : P \times P \longrightarrow \langle e^{\pi i/(n+1)^2} \rangle \tag{1.2}$$

corresponding to the extension, then one has

$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha)^{-1} = (-1)^{\langle \alpha, \beta \rangle} \quad \text{for } \alpha, \beta \in Q. \tag{1.3}$$

We adopt the notation

$$c(\lambda, \mu) := \varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda)^{-1} \quad \text{for } \lambda, \mu \in P; \tag{1.4}$$

this is the bimultiplicative alternating commutator map of the central extension (cf. [22, Ch. 5]; [13, Ch. 2 and 13]).

The affine Lie algebra $\hat{\mathfrak{g}}$ (of type $A_n^{(1)}$) is the infinite-dimensional Lie algebra $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ with bracket given by

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + \langle x, y \rangle\, r\, \delta_{r+s,0}\, c, \tag{1.5}$$

where $x, y \in \mathfrak{g}$, $r, s \in \mathbb{Z}$ and c is central. One also needs the grading operator $D := -t\,\mathrm{d}/\mathrm{d}t$. Denote the set of simple (positive) roots of $\hat{\mathfrak{g}}$ by $\hat{\Pi} := \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset (\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}D)^*$. The usual extensions of $\langle\cdot,\cdot\rangle$ on $\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}D$ and on $(\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}D)^*$ will be denoted by the same symbol (we take $\langle c, D\rangle = -1$).

We shall often be working with the nilpotent subalgebras $\mathfrak{n}_\pm := \mathrm{span}_{\mathbb{C}}\{x_\alpha \mid \alpha \in \Delta_\pm\} \subset \mathfrak{g}$ and the corresponding subalgebra $\bar{\mathfrak{n}}_\pm := \mathfrak{n}_\pm \otimes \mathbb{C}[t, t^{-1}]$ (without c) of $\hat{\mathfrak{g}}$ (the affinization of $\mathfrak{n}_\pm$ is denoted $\hat{\mathfrak{n}}_\pm := \mathfrak{n}_\pm \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \subset \hat{\mathfrak{g}}$). For the sake of building our basis we shall need the one-dimensional subalgebras $\mathfrak{n}_\beta := \mathbb{C}x_\beta$, $\beta \in \Delta_\pm$, of $\mathfrak{g}$ and the respective abelian subalgebras $\bar{\mathfrak{n}}_\beta := \mathfrak{n}_\beta \otimes \mathbb{C}[t, t^{-1}]$ of $\hat{\mathfrak{g}}$ (as opposed to the affinizations $\hat{\mathfrak{n}}_\beta := \mathfrak{n}_\beta \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$). A crucial ingredient of the vertex operator construction of basic $\hat{\mathfrak{g}}$-modules is another affine Lie algebra: the subalgebra $\hat{\mathfrak{h}} := \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ of $\hat{\mathfrak{g}}$.

Recall that for a fixed level $k \in \mathbb{Z}_+$ (the scalar by which c will act on a module), the set of dominant integral weights of $\hat{\mathfrak{g}}$ is $\{\sum_{j=0}^n k_j \hat{\Lambda}_j \mid k_j \in \mathbb{N},\ \sum_{j=0}^n k_j = k\}$, where $\hat{\Lambda}_i \in (\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}D)^*$, $i = 0, \ldots, n$, are the fundamental weights of $\hat{\mathfrak{g}}$, i.e., $\langle \hat{\Lambda}_i, \alpha_j\rangle = \delta_{ij}$ and $\hat{\Lambda}_i(D) = 0$, $i, j = 0, \ldots, n$. The notation $L(\hat{\Lambda})$ will signify the standard (integrable irreducible highest weight) $\hat{\mathfrak{g}}$-module of level k and highest weight $\hat{\Lambda} = \sum_{j=0}^n k_j \hat{\Lambda}_j$, where $\sum_{j=0}^n k_j = k$.

Let $v(\hat{\Lambda})$ be a highest weight vector of $L(\hat{\Lambda})$. Following Feigin–Stoyanovsky [16], we define the *principal subspace*

$$W(\hat{\Lambda}) := U(\hat{\mathfrak{n}}_+) \cdot v(\hat{\Lambda}) = U(\bar{\mathfrak{n}}_+) \cdot v(\hat{\Lambda}), \tag{1.6}$$

where $U(\cdot)$ always denotes universal enveloping algebra (similarly, $S(\cdot)$ always denotes a symmetric algebra). The principal subspace is defined the same way for any highest weight module.

For $k \in \mathbb{C}^\times$, consider the induced $\hat{\mathfrak{h}}$-module

$$M(k) := U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c)} \mathbb{C}, \tag{1.7}$$

with $\mathfrak{h} \otimes \mathbb{C}[t]$ acting trivially on $\mathbb{C}$ and c acting as k. It is naturally isomorphic as a vector space to the symmetric algebra $S(\hat{\mathfrak{h}}^-)$, where $\hat{\mathfrak{h}}^- := \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ (similarly, set $\hat{\mathfrak{h}}^+ := \mathfrak{h} \otimes t\mathbb{C}[t]$).

For a given $s \in \mathbb{Z}_+$, we shall be referring to the "difference two at distance s" condition, defined as follows: suppose we have a sequence of integers, nonincreasing from right to left,

$$m_r \leq \cdots \leq m_2 \leq m_1, \tag{1.8}$$

which are indexed according to their place in the sequence, counted from right to left (one can think of such sequence as a partition of the sum of its entries). The sequence is said to satisfy the "difference two at distance s" condition, $s \in \mathbb{Z}_+$, $s < r$, if

$$m_{t+s} \le m_t - 2, \quad 1 \le t \le r - s. \tag{1.9}$$

The following strict linear (lexicographic) ordering "$<$" and strict partial ordering "$\prec$" (called also "multidimensional" in [41]) will be largely used in our arguments: For given $r_n, \ldots, r_1 \in \mathbb{Z}_+$, $\sum_{i=1}^n r_i = r$, consider color-ordered sequences of r_n integers of "color" n, ..., r_1 integers of color 1:

$$m_r \le \cdots \le m_{\sum_{i=1}^{n-1} r_i + 1}, m_{\sum_{i=1}^{n-1} r_i} \le \cdots \le m_{r_1+1}, m_{r_1} \le \cdots \le m_1, \tag{1.10}$$

such that only the entries of the same color are nonincreasing from right to left. For two such sequences, we write

$$(m_r, \ldots, m_2, m_1) < (m'_r, \ldots, m'_2, m'_1) \tag{1.11}$$

if there exists $s \in \mathbb{Z}_+$, $1 \le s \le r$, such that $m_1 = m'_1$, $m_2 = m'_2, \ldots, m_{s-1} = m'_{s-1}$ and $m_s < m'_s$. On the other hand, we write

$$(m_r, \ldots, m_2, m_1) \prec (m'_r, \ldots, m'_2, m'_1) \tag{1.12}$$

if for every s, $1 \le s \le r$, one has $m_s + \cdots + m_2 + m_1 \le m'_s + \cdots + m'_2 + m'_1$ and for at least one such s, this inequality is strict.

It is easy to see that $a \prec b$ implies $a < b$ but not vice versa.

We shall also encounter more general situations when an additional characteristic "charge" is assigned to the entries of our sequences and (1.10) is generalized as follows: the "monochromatic" segments are broken into subsegments of entries of the same charge so that entries of larger charge are always on the right-hand side of entries of smaller charge (of the same color) and only the entries of the same charge and color are ordered (nonincreasing from right to left).

2. Homogeneous vertex operator construction of basic (level 1 standard) modules

Since most of our considerations refer to the vertex operator algebra approach to the basic $\hat{\mathfrak{g}}$-modules (the standard $\hat{\mathfrak{g}}$-modules of level 1), we shall briefly sketch it in this section. We work in the setting of [22] and [13], to which we refer for more details; see also [6, 21, 31, 55].

Consider the tensor product vector spaces $V_Q := M(1) \otimes \mathbb{C}[Q]$, $V_P := M(1) \otimes \mathbb{C}[P]$. We shall be using independent commuting formal variables $z, z_0, z_1, z_2, \ldots$. For any vector space V we denote by $V[[z]]$ the space of all (possibly infinite) formal series of nonnegative integral powers of z with coefficients in V. Similarly, we denote by $V\{z\}$ the space of all (possibly infinite in both directions) formal series of rational powers of z with coefficients in V. Recall that V_Q has a natural structure of simple vertex operator algebra (VOA) and that V_P is a module for this VOA through a linear map, which we define on all of V_P, rather than just V_Q (cf. [13]):

$$\begin{array}{rcl} V_P & \longrightarrow & (\operatorname{End} V_P)\{z\} \\ v & \longmapsto & Y(v,z) \end{array}$$

given by

$$Y(1\otimes e^{\lambda},z) := \exp\left\{\sum_{n\geq 1} h_{\lambda}(-n)\frac{z^n}{n}\right\} \exp\left\{\sum_{n\geq 1} h_{\lambda}(n)\frac{z^{-n}}{-n}\right\} \otimes e^{\lambda}\, z^{h_{\lambda}}\, \varepsilon_{\lambda} \tag{2.1}$$

for $\lambda \in P$ and by

$$\begin{aligned} & Y\left(\prod_{i=1}^{l} h_i(-n_i)\otimes e^{\lambda},\, z\right) \\ & := : \prod_{i=1}^{l}\left[\frac{1}{(n_i-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_i-1} h_i(z)\right] Y(1\otimes e^{\lambda},z) \end{aligned} \tag{2.2}$$

for a generic homogeneous vector ($h_i \in \mathfrak{h}$, $n_i \in \mathbb{Z}_+$, $\lambda \in P$), where the following notations are used:

$$h(m) := h\otimes t^m \quad \text{for all } h\in\mathfrak{h},\ m\in\mathbb{Z},$$

$$z^{h_{\lambda}}e^{\mu} := z^{\langle\lambda,\mu\rangle}e^{\mu} \quad \text{for all } \lambda,\mu\in P,$$

$$\varepsilon_{\lambda}e^{\mu} := \varepsilon(\lambda,\mu)e^{\mu} \quad \text{for all } \lambda,\mu\in P,$$

$$h(z) := \sum_{m\in\mathbb{Z}} (h(m)\otimes 1)z^{-m-1} \quad \text{for all } h\in\mathfrak{h},$$

and $:\ \cdot\ :$ is a normal ordering procedure, which signifies that the enclosed expression is to be reordered if necessary so that all the operators $h(m)$ ($h \in \mathfrak{h}$, $m < 0$) are to be placed to the left of all the operators $h(m)$ ($h \in \mathfrak{h}$, $m \geq 0$). Abusing notation, we shall often write e^{λ} instead of $1 \otimes e^{\lambda}$ and $h(m)$ instead of $h(m)\otimes 1$.

Recall the following classical interpretation of V_P as a $\hat{\mathfrak{g}}$-module ([21, 55]): Let $x_\alpha \otimes t^m$, $\alpha \in \Delta$, $m \in \mathbb{Z}$ act on V_P as $x_\alpha(m)$, where $x_\alpha(m)$ is defined as a coefficient of the vertex operator $Y(e^\alpha,z)$:

$$\sum_{m\in\mathbb{Z}} x_\alpha(m)z^{-m-1} := Y(e^{\alpha},z); \tag{2.3}$$

let $h\otimes t^m$, $h \in \mathfrak{h}$, $m \in \mathbb{Z}$ act on V_P as $h(m)$ (recall that this is our abbreviated notation for the operator $h(m)\otimes 1$) and finally, let the central element c act as the identity operator. Then this action endows V_Q and $V_Q e^{\Lambda_j}$ for $j = 1,\dots,n$ with the structure of level 1 standard $\hat{\mathfrak{g}}$-modules with highest weight vectors $v(\hat{\Lambda}_0) := 1\otimes 1$ and $v(\hat{\Lambda}_j) := 1\otimes e^{\Lambda_j}$, $j = 1,\dots,n$, respectively. In other words $V_Q \cong L(\hat{\Lambda}_0)$, $V_Q e^{\Lambda_j} \cong L(\hat{\Lambda}_j)$ for

$j = 1,\dots,n$ and therefore $V_P \cong \oplus_{j=0}^{n} L(\hat{\Lambda}_j)$. Note that from the very definitions (2.1) and (2.3), one has

$$x_\alpha(m)e^{\lambda} = e^{\lambda}x_\alpha(m + \langle\lambda,\alpha\rangle), \quad \lambda \in P, \alpha \in \Delta. \tag{2.4}$$

Adopting the standard notation

$$E^{\pm}(h,z) := \exp\left(\sum_{m\geq 1} h(\pm m)\frac{z^{\mp m}}{\pm m}\right) \tag{2.5}$$

for $h \in \mathfrak{h}$, one can rewrite (2.1) as

$$Y(e^{\lambda},z) = E^{-}(-h_\lambda,z)E^{+}(-h_\lambda,z) \otimes e^{\lambda}z^{h_\lambda}\varepsilon_\lambda. \tag{2.6}$$

Recall that the commutation relations among these vertex operators as well as some nice properties of their products (see the next section) follow from the "commutation relation" of E^+ and E^- (cf. [22, Ch. 4])

$$E^{+}(-h_\lambda,z_2)E^{-}(-h_\mu,z_1) = \left(1 - \frac{z_1}{z_2}\right)^{\langle\lambda,\mu\rangle} E^{-}(-h_\mu,z_1)E^{+}(-h_\lambda,z_2), \tag{2.7}$$

where $\lambda, \mu \in P$ and the binomial expression is to be expanded in nonnegative powers of z_1/z_2. As a corollary,

$$\begin{aligned} &E^{+}(-h_\lambda,z_2)E^{-}(-h_\mu,z_1) \otimes (e^{\lambda}z_2^{h_\lambda}\varepsilon_\lambda)(e^{\mu}z_1^{h_\mu}\varepsilon_\mu) \\ &\quad = \mathrm{const}(z_2 - z_1)^{\langle\lambda,\mu\rangle}E^{-}(-h_\mu,z_1)E^{+}(-h_\lambda,z_2) \otimes e^{\lambda+\mu}z_2^{h_\lambda}z_1^{h_\mu}\varepsilon_\lambda\varepsilon_\mu, \end{aligned} \tag{2.8}$$

where const $\in \mathbb{C}^{\times}$.

There is a Jacobi identity for the operators $Y(v,z)$, $v \in V_P$ (see [13, Ch. 5]), and we shall be particularly interested in the following case (see [13] formula (12.5)): For $\lambda \in Q, \mu \in P; u^*, v^* \in M(1)$; $u := u^* \otimes e^{\lambda}, v := v^* \otimes e^{\mu}$, one has

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v,z_2) - (-1)^{\langle\lambda,\mu\rangle}c(\lambda,\mu)z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1) \\ &\quad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v,z_2), \end{aligned} \tag{2.9}$$

where $\delta(z) := \sum_{m\in\mathbb{Z}} z^m$ is the usual formal delta function and the binomial expressions are to be expanded in nonnegative integral powers of the second variable.

Note that for $\mu \in Q$, we have $(-1)^{\langle\lambda,\mu\rangle}c(\lambda,\mu) = 1$, giving the ordinary Jacobi identity for the vertex operator algebra V_Q and for its action on the irreducible modules $V_Q e^{\Lambda_j}$, $j = 1,\dots,n$. In order to get rid of the numerical factor $(-1)^{\langle\lambda,\mu\rangle}c(\lambda,\mu)$ even if $\mu \notin Q$, we first note that the fundamental weights $\Lambda_1,\dots,\Lambda_n$, which are all minuscule, constitute a set of representatives for the nontrivial cosets of Q in P. As in [13, formula (12.3)], for $\mu \in \Lambda_j + Q$, $j = 1,\dots,n$, we replace $Y(v,z)$ by $\mathscr{Y}(v,z) := Y(v,z)e^{i\pi h_{\Lambda_j}}c(\cdot,\Lambda_j)$ (where the operators $e^{i\pi h_{\Lambda_j}}$ and $c(\cdot,\Lambda_j)$ are defined in the obvious ways). For $\mu \in Q$ we simply set $\mathscr{Y}(v,z) := Y(v,z)$. This gives us a linear map $v \mapsto \mathscr{Y}(v,z)$ from V_P to $(\mathrm{End} V_P)\{z\}$, and by Proposition 12.2 of [13], we have the ordinary Jacobi identity

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)\mathscr{Y}(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) \mathscr{Y}(v,z_2)Y(u,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \mathscr{Y}(Y(u,z_0)v,z_2) \tag{2.10}$$

(see [13, formula (12.8)]). This identity, together with certain other natural conditions (again see [13, Proposition 12.2]), guarantees that $\mathscr{Y}(\cdot,z)$ defines an intertwining operator in the sense of [20]. More precisely, let us write $\Lambda_0 := 0$ for convenience. Then for $\mu \in \Lambda_j + Q$, $j = 0,1,\ldots,n$, $\mathscr{Y}(v,z)$ (with v as above) defines an intertwining operator of type $\left[\begin{smallmatrix} l \\ ji \end{smallmatrix}\right]$, $l \equiv (i+j) \bmod (n+1)$, since the correspondence $j \mapsto \Lambda_j$ defines an isomorphism from the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ to P/Q (we are indexing the irreducible V_Q-modules $V_Q e^{\Lambda_j}$ by the integers $j \bmod (n+1)$); cf. [13, Ch. 12]. In particular, one has a map

$$\mathscr{Y}(e^{\Lambda_j},z) : L(\hat{\Lambda}_i) \to L(\hat{\Lambda}_l)\{z\}$$

for $l \equiv (i+j) \bmod (n+1)$.

Now in (2.9) make the specializations $u := e^{\alpha}$, $\alpha \in \Delta_+$ and $v := e^{\Lambda_j}$ for any $j \in \{1,\ldots,n\}$. Notice that from the very definition (2.1), one has $Y(e^{\alpha},z_0)e^{\Lambda_j} \in V_P[[z_0]]$ and therefore, taking Res_{z_0} of (2.9), we simply get

$$[Y(e^{\alpha},z_1)\,,\, \mathscr{Y}(e^{\Lambda_j},z_2)] = 0 \quad \text{for } \alpha \in \Delta_+,\ j = 1,\ldots,n. \tag{2.11}$$

This seemingly innocent commutativity, which asserts that the coefficients of the series $\mathscr{Y}(e^{\Lambda_j},z)$ are $\bar{\mathfrak{n}}_+$-module maps, has deep implications for the representation theory of $\hat{\mathfrak{g}}$: One is tempted to interpret it as a device to "lift" relations (or constraints, as physicists would say) between operators from $U(\hat{\mathfrak{n}}_+)$ acting on $L(\hat{\Lambda}_i)$ to analogous relations among these operators, but acting on $L(\hat{\Lambda}_l)$. In particular, when questions of basis are concerned, it can be very advantageous to treat all the simple modules at a given level simultaneously. We shall later demonstrate fruitful applications of this strategy (Theorem 4.2).

We close this Section with the definition of a projection needed only for Part II [25]: Since the grading $\mathbb{C}[Q] = \coprod_{\beta \in Q} \mathbb{C}e^{\beta}$ of the group lattice $\mathbb{C}[Q]$ induces a grading of the whole basic module $L(\hat{\Lambda}_j) = M(1) \otimes \mathbb{C}[Q]e^{\Lambda_j}$, $0 \le j \le n$, we can define

$$\pi_{U(\hat{\mathfrak{h}}^-)\cdot v(\hat{\Lambda}_j)} : L(\hat{\Lambda}_j) \to U(\hat{\mathfrak{h}}^-) \cdot v(\hat{\Lambda}_j) = M(1) \otimes e^{\Lambda_j} \cong M(1) \tag{2.12}$$

to be the corresponding projection on the homogeneous subspace $U(\hat{\mathfrak{h}}^-) \cdot v(\hat{\Lambda}_j)$. Using the grading $U(\hat{\mathfrak{h}}^-) = \coprod_{m \in \mathbb{N}} U^m(\hat{\mathfrak{h}}^-)$ by symmetric powers $U^m(\hat{\mathfrak{h}}^-) = S^m(\hat{\mathfrak{h}}^-)$, we can go one step further and define for every $m \in \mathbb{N}$ the coresponding projection

$$\pi_{U^m(\hat{\mathfrak{h}}^-)\cdot v(\hat{\Lambda}_j)} : L(\hat{\Lambda}_j) \to U^m(\hat{\mathfrak{h}}^-) \cdot v(\hat{\Lambda}_j). \tag{2.13}$$

3. Quasi-particles

We begin with the choice of a special subspace of $U(\bar{\mathfrak{n}}_+)$ whose appropriate completions will contain all the basis-generating operators considered below. Set U to be the subspace which is the ordered product of universal enveloping algebras

$$U := U(\bar{\mathfrak{n}}_{\alpha_n})U(\bar{\mathfrak{n}}_{\alpha_{n-1}})\cdots U(\bar{\mathfrak{n}}_{\alpha_1}). \tag{3.1}$$

This is a product of subalgebras of $U(\hat{\mathfrak{g}})$, and by the Poincaré–Birkhoff–Witt theorem, it is linearly isomorphic to the tensor product $U(\bar{\mathfrak{n}}_{\alpha_n}) \otimes U(\bar{\mathfrak{n}}_{\alpha_{n-1}}) \otimes \cdots \otimes U(\bar{\mathfrak{n}}_{\alpha_1})$.

Fix a level $k \in \mathbb{Z}_+$ and let $\hat{\Lambda}$ be a dominant integral highest weight of this level, $\hat{\Lambda} = k\hat{\Lambda}_0 + \Lambda$, $\Lambda \in P$. Denote the action of $x_\alpha \otimes t^m \in \hat{\mathfrak{g}}$, $\alpha \in \Delta$, $m \in \mathbb{Z}$, on the standard module $L(\hat{\Lambda})$ by $x_\alpha(m)$ (in the previous section we have constructed explicitly $x_\alpha(m)$ for basic modules). Denote the corresponding generating function (vertex operator) by $X_\alpha(z)$, i.e.,

$$X_\alpha(z) := \sum_{m\in\mathbb{Z}} x_\alpha(m)z^{-m-1}, \quad \alpha \in \Delta, \tag{3.2}$$

where z is a formal variable. For example, for the explicit realization of basic modules given in the previous section, one obviously has $X_\alpha(z) = Y(e^\alpha, z)$ (cf. (2.3)). More generally, a standard module at any level $k \in \mathbb{Z}_+$ can be constructed explicitly as a subspace of the tensor product of k such basic modules: In this case we simply set $X_\alpha(z) := \Delta^{k-1}(Y(e^\alpha, z))$, where Δ^{k-1} is the $(k-1)$-iterate of the standard coproduct Δ, and then our module is generated by the tensor product of the k level one highest weight vectors (no confusion can arise from the fact that Δ denotes also the set of $\mathfrak{g}$-roots).

Let us quickly show that the principal subspace $W(\hat{\Lambda})$ of the standard module $L(\hat{\Lambda})$ is indeed generated by operators in U acting on the highest weight vector $v(\hat{\Lambda})$ (this is true for any highest weight $\hat{\mathfrak{g}}$-module).

Lemma 3.1. *One has*

$$W(\hat{\Lambda}) := U(\bar{\mathfrak{n}}_+) \cdot v(\hat{\Lambda}) = U \cdot v(\hat{\Lambda}).$$

Proof. Observe that $W(\hat{\Lambda})$ is spanned by

$$\{b \cdot v(\hat{\Lambda}) | b \in U(\bar{\mathfrak{n}}_{\beta_r})\cdots U(\bar{\mathfrak{n}}_{\beta_1});\ \beta_1, \ldots, \beta_r \in \Pi\},$$

since every x_β, $\beta \in \Delta_+$, can be expressed as a bracket of x_{α_i}, $1 \le i \le n$. We want to show that the spanning property will still hold even if we order this product of universal enveloping algebras. In other words, we have to find a way to change the order in the product $x_{\alpha_i}(l)x_{\alpha_{i+1}}(m)$, $1 \le i \le n-1$, when acting on a given vector $v \in W(\hat{\Lambda})$, possibly at the expense of changing the indices. But this is easy, because for every $m \in \mathbb{Z}$, there exists $N \gg 0$ such that $x_{\alpha_{i+1}}(m+N) \cdot v = 0$ and therefore

$$\begin{aligned} x_{\alpha_i}(l)x_{\alpha_{i+1}}(m) \cdot v &= \pm x_{\alpha_i+\alpha_{i+1}}(l+m) \cdot v + x_{\alpha_{i+1}}(m)x_{\alpha_i}(l) \cdot v \\ &= -x_{\alpha_{i+1}}(m+N)x_{\alpha_i}(l-N) \cdot v + x_{\alpha_{i+1}}(m)x_{\alpha_i}(l) \cdot v. \quad \square \end{aligned}$$

Following physicists' terminology (cf. e.g. [11]), we shall say that an operator $x_{\alpha_i}(m) \in \bar{\mathfrak{n}}_{\alpha_i}$ represents a *quasi-particle of color* i *and charge* 1 (the eigenvalue $-m$ of the scaling operator D under the adjoint action is called the *energy* of our quasi-particle). Moreover, a monomial from $U(\bar{\mathfrak{n}}_{\alpha_n})U(\bar{\mathfrak{n}}_{\alpha_{n-1}})\cdots U(\bar{\mathfrak{n}}_{\alpha_1})$ is *of color-type* $(r_n; r_{n-1}; ...; r_1)$ if it carries charge r_n with color n, charge r_{n-1} with color $n-1$ and so on (we are using the grading of each $U(\bar{\mathfrak{n}}_{\alpha_i}) = S(\bar{\mathfrak{n}}_{\alpha_i})$ by symmetric powers and taking the tensor product grading). We thus obtain a "color-type" gradation of the whole vector space $U := U(\bar{\mathfrak{n}}_{\alpha_n})U(\bar{\mathfrak{n}}_{\alpha_{n-1}})\cdots U(\bar{\mathfrak{n}}_{\alpha_1})$:

$$U = \coprod_{r_n,\dots,r_1 \geq 0} U_{(r_n;\dots;r_1)}. \tag{3.3}$$

Note that for every (dominant integral) highest weight $\hat{\Lambda}$, the principal subspace $W(\hat{\Lambda})$ also has a color-type gradation (compatible with the $\mathfrak{h}$-gradation of $L(\hat{\Lambda})$):

$$W(\hat{\Lambda}) = \coprod_{r_n,\dots,r_1 \geq 0} W(\hat{\Lambda})_{(r_n;\dots;r_1)}, \tag{3.4}$$

where

$$W(\hat{\Lambda})_{(r_n,\dots;r_1)} := W_{\Lambda + \sum_{i=1}^{n} r_i\alpha_i}(\hat{\Lambda}) \tag{3.5}$$

is the weight subspace of weight $\Lambda + \sum_{i=1}^{n} r_i\alpha_i \in P$.

When a given monomial is not monochromatic, it is somewhat convenient to have the color-type implicitly encoded in its notation. We shall do this by adding a second subscript i to all the entries associated with a quasi-particle of color i and having the other subscript enumerating quasi-particles of a given color only (as opposed to using a single subscript as in (1.10) for example).

Unless stated otherwise, we shall assume that a product of (commuting) quasi-particles of the same color i (and charge 1) has its quasi-particle indices nonincreasing from right to left. With this convention in mind, note that the set of quasi-particle monomials from U of a given color-type $(r_n; \dots; r_1)$ is linearly ordered by "$<$" if the definition (1.11) is applied to the respective index sequences. Moreover, for two such quasi-particle monomials b and b', we write $b \prec b'$ if $b < b'$ and in addition,

$$(m_n; \dots; m_1) \prec (m_n'; \dots; m_1')$$

(cf. (1.12)), where m_i (resp. m_i'), $1 \leq i \leq n$, is the sum of indices of all the quasi-particles of color i in b (resp. b').

The partial ordering "$\prec$" will be pivotal in our spanning arguments because it has the nice property that $\prec$-intervals are *finite* (which is not true for its linear extension "$<$"; it was an idea of J. Lepowsky to employ the multidimensional ordering "$\prec$", fundamental in [41], in the current setting). The lexicographic ordering "$<$" has this property only if the index-sum for every single color is held fixed (because quasi-particles of different colors do not commute).

One can naturally generalize the above concept of quasi-particle of charge 1 and define quasi-particles of arbitrary charge $r \in \mathbb{Z}_+$ as follows:

Definition 3.1. For given i, $1 \le i \le n$, and $r \in \mathbb{Z}_+$, $m \in \mathbb{Z}$, define

$$\begin{aligned} x_{r\alpha_i}(m) &:= \sum_{\substack{m_r,\dots,m_1 \in \mathbb{Z} \\ m_r+\cdots+m_1=m}} x_{\alpha_i}(m_r)\cdots x_{\alpha_i}(m_1) \\ &= \mathrm{Res}_z\{z^{m+r-1}\underbrace{X_{\alpha_i}(z)\cdots X_{\alpha_i}(z)}_{r\ factors}\} \end{aligned} \tag{3.6}$$

(the indices $m_r,\dots,m_1$ in the above multisum are not ordered!). We call $x_{r\alpha_i}(m)$ a *quasi-particle of color i and charge r* (the eigenvalue $-m$ of the scaling operator D under the adjoint action is as usual the *energy* of our quasi-particle). Abusing language, we shall say that $x_{r\alpha_i}(m)$ is from $U(\bar{\mathfrak{n}}_{\alpha_i})$ because our quasi-particles will always act on highest weight modules, in which case, the sum above is finite (note that this sum is infinite in general, i.e., the quasi-particles lie in an appropriate completion of $U(\bar{\mathfrak{n}}_{\alpha_i})$.

Note that a quasi-particle of charge r can be thought of as a cluster of r quasi-particles of charge one confined in such a way that only the total index-sum (i.e., the energy of the cluster with a minus sign) is "measurable", while the individual quasi-particle indices run through $\mathbb{Z}$. Nevertheless, just like the quasi-particles of charge 1, the quasi-particles of charge r are coefficients of certain vertex operators: If we set

$$X_{r\alpha_i}(z) := \underbrace{X_{\alpha_i}(z)\cdots X_{\alpha_i}(z)}_{r\ factors} = \sum_{m\in\mathbb{Z}} x_{r\alpha_i}(m)z^{-m-r}, \tag{3.7}$$

(cf. (3.2)), one can show that $X_{r\alpha_i}(z)$ is the vertex operator corresponding to the vector $x_{\alpha_i}(-1)^r \cdot v(k\hat{\Lambda}_0)$, where $v(k\hat{\Lambda}_0)$ is the vacuum highest weight vector at the chosen level $k \in \mathbb{Z}_+$. In other words,

$$X_{r\alpha_i}(z) = Y(x_{\alpha_i}(-1)^r \cdot v(k\hat{\Lambda}_0), z), \tag{3.8}$$

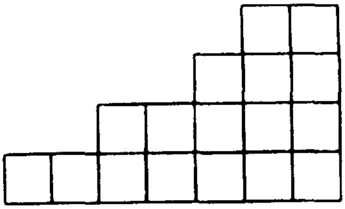
(cf. [13, Proposition 13.16]).

Unless stated otherwise, we shall assume that a product of (commuting) quasi-particles of the same color and charge has its indices nonincreasing from right to left. Moreover, a product of (commuting) quasi-particles of the same color but different charges will have its charges nonincreasing from right to left (not surprisingly, quasi-particles of the same color but different charges will behave like different objects; cf. Section 5).

Pick a Young diagram (partition)

$$r^{(1)} \ge r^{(2)} \ge \cdots \ge r^{(K)} > 0, \quad \sum_{t=1}^{K} r^{(t)} = r,\ K \in \mathbb{Z}_+, \tag{3.9}$$

pictured as follows:



(the bottom row has $r^{(1)}$ squares, the second row has $r^{(2)}$ squares, ..., the top Kth row has $r^{(K)}$ squares). Let the dual Young diagram (in reversed order) be

$$0 < n_{r^{(1)}} \le \cdots \le n_2 \le n_1 = K, \quad \sum_{p=1}^{r^{(1)}} n_p = r, \tag{3.10}$$

(i.e., the rightmost column has n_1 boxes, the second-from-right column has n_2 boxes, ..., the leftmost column has $n_{r^{(1)}}$ boxes).

Fix a color i, $1 \le i \le n$. We say that a monochromatic quasi-particle monomial is of *charge-type* $(n_{r^{(1)}}, \ldots, n_1)$ and of *dual-charge-type* $(r^{(1)}, \ldots, r^{(K)})$ if it is built (in the obvious sense) out of $r^{(1)} - r^{(2)}$ quasi-particles of charge 1, $r^{(2)} - r^{(3)}$ quasi-particles of charge 2, ..., $r^{(K)}$ quasi-particles of charge K. In other words, each quasi-particles of charge r is represented by a column of height r in the respective Young diagram.

The same terminology of course applies to the corresponding generating functions (this is closer to the setting in which Feigin and Stoyanovsky talk about clusters; cf. [16, Theorem 2.7.1]): We say that

$$X_{n_{r^{(1)}}\alpha_i}(z_{r^{(1)}}) \cdots X_{n_1\alpha_i}(z_1) \tag{3.11}$$

is of *charge-type* $(n_{r^{(1)}}, \ldots, n_1)$ and of *dual-charge-type* $(r^{(1)}, \ldots, r^{(K)})$.

In complete analogy with the charge one picture, we can extend our dictionary to multi-colored quasi-particle monomials from U

$$b := b_n \cdots b_2 b_1, \tag{3.12}$$

where b_i, $1 \le i \le n$, is a monochromatic quasi-particle monomial of color i (simply order the colors and add a subscript i to the entries corresponding to the color i). It is clear what it means for the quasi-particle monomial (3.12) to be of *color-charge-type*

$$(n_{r_n^{(1)},n}, \ldots, n_{1,n}; \ldots; n_{r_1^{(1)},1}, \ldots, n_{1,1}), \tag{3.13}$$

where

$$0 < n_{r^{(1)},i} \le \cdots \le n_{2,i} \le n_{1,i} \le K, \quad \sum_{p=1}^{r_i^{(1)}} n_{p,i} = r_i, \ 1 \le i \le n,$$

of *color-dual-charge-type*

$$(r_n^{(1)}, \ldots, r_n^{(K)}; \ldots; r_1^{(1)}, \ldots, r_1^{(K)}), \tag{3.14}$$

where

$$r_i^{(1)} \geq r_i^{(2)} \geq \cdots \geq r_i^{(K)} \geq 0, \quad \sum_{t=1}^{K} r_i^{(t)} = r_i, \; K \in \mathbb{Z}_+, \; 1 \leq i \leq n$$

(we shall sometimes be writing only the leftmost and the rightmost entries inside the parentheses) and of *color-type* $(r_n; \ldots; r_1)$. It shall also say that the corresponding generating function

$$X_{n_{r_n^{(1)},n}\alpha_n}(z_{n_{r_n^{(1)},n}}) \cdots X_{n_{1,1}\alpha_1}(z_{1,1}) \tag{3.15}$$

is of the above color-charge-type and color-dual-charge-type.

Finally, we can naturally extend both the linear ordering "$<$" and the partial ordering "$\prec$" to the set of multi-colored higher-charge quasi-particle monomials of given color-type $(r_n; \ldots; r_1)$. The lexicographic ordering "$<$" is defined as follows: First apply definition (1.11) to the *color-charge-types* of the two monomials b and b' (rather than to their index sequences!); if the color-charge-types are the same, apply (1.11) to the index sequences of the two monomials. In complete analogy with the charge-one situation, the partial ordering "$\prec$" is defined by "restricting" the lexicographic ordering "$<$" as follows: We write $b \prec b'$ if $b < b'$ and in addition,

$$(m_n; \ldots; m_1) \prec (m'_n; \ldots; m'_1)$$

(cf. (1.12)), where m_i (resp., m'_i), $1 \leq i \leq n$, is the sum of indices of all the quasi-particles of color i in b (resp., b').

One should be aware that throughout the rest of the paper, the lexicographic ordering in the set of color-charge-types (for fixed color-type) will be truncated from above in a very special way. Namely, formula (3.8) and the null vector identity

$$x_{\alpha_i}(-1)^{k+1} \cdot v(k\hat{\Lambda}_0) = 0, \tag{3.16}$$

(where $v(k\hat{\Lambda}_0)$ is the vacuum highest weight vector at a given level $k \in \mathbb{Z}_+$), imply that

$$X_{(k+1)\alpha_i}(z) = \underbrace{X_{\alpha_i}(z) \cdots X_{\alpha_i}(z)}_{k+1 \; factors} = Y(x_{\alpha_i}(-1)^{k+1} \cdot v(k\hat{\Lambda}_0), z) = 0, \tag{3.17}$$

i.e., all the quasi-particles of charge greater than k are zero when acting on level k standard modules (this was the main idea on which [39] was based). "Annihilating" relations like this are not enough for constructing a quasi-particle basis at levels $k > 1$. We shall have to employ in addition the following (obvious from (3.7) and (3.8)) relations which express quasi-particle monomials of a given (color)-charge-type through quasi-particle monomials of the same color-type but of greater in the ordering "$<$" (color)-charge-types: for $0 < s \leq s' \leq k$, one has

$$\begin{aligned} X_{s\alpha_i}(z)X_{s'\alpha_i}(z) &= X_{(s-1)\alpha_i}(z)X_{(s'+1)\alpha_i}(z) \\ &= \cdots = X_{\alpha_i}(z)X_{(s'+s-1)\alpha_i}(z) = X_{(s'+s)\alpha_i}(z). \end{aligned} \tag{3.18}$$

These are s independent relations for monochromatic quasi-particle monomials of charge-type (s, s'), $0 < s \leq s' \leq k$, which express them through quasi-particle monomials of greater charge-types. One can actually rewrite these relations among vertex operators as (equally obvious) equivalent relations among the corresponding vectors, cf. (3.8):

$$\begin{aligned} x_{s\alpha_i}(-s)x_{s'\alpha_i}(-s') \cdot v(k\hat{\Lambda}_0) &= x_{(s-1)\alpha_i}(-(s-1))x_{(s'+1)\alpha_i}(-(s'+1)) \cdot v(k\hat{\Lambda}_0) \\ &= \cdots = x_{\alpha_i}(-1)x_{(s'+s-1)\alpha_i}(-(s'+s-1) \cdot v(k\hat{\Lambda}_0) \\ &= x_{(s'+s)\alpha_i}(-(s'+s)) \cdot v(k\hat{\Lambda}_0) \\ &= \underbrace{x_{\alpha_i}(-1)x_{\alpha_i}(-1)\cdots x_{\alpha_i}(-1)}_{s'+s\ factors} \cdot v(k\hat{\Lambda}_0). \end{aligned} \tag{3.19}$$

Another series of fundamental relations, independent from the ones above, is the following: for $0 < s < s' \leq k$, one has

$$\frac{1}{s}\left(\frac{\mathrm{d}}{\mathrm{d}z}X_{s\alpha_i}(z)\right)X_{s'\alpha_i}(z) = \frac{1}{s'}X_{s\alpha_i}(z)\left(\frac{\mathrm{d}}{\mathrm{d}z}X_{s'\alpha_i}(z)\right), \tag{3.20}$$

which follows from the corresponding vector relation

$$\begin{aligned} \frac{1}{s}x_{s\alpha_i}(-s-1)x_{s'\alpha_i}(-s') \cdot v(k\hat{\Lambda}_0) &= \frac{1}{s'}x_{s\alpha_i}(-s)x_{s'\alpha_i}(-s'-1) \cdot v(k\hat{\Lambda}_0) \\ &= \underbrace{x_{\alpha_i}(-2)x_{\alpha_i}(-1)\cdots x_{\alpha_i}(-1)}_{s'+s \text{ factors}} \cdot v(k\hat{\Lambda}_0). \end{aligned} \tag{3.21}$$

Note that the relation (3.20) is trivial for $s = s'$. Combining (3.20) with (3.18), we can replace (3.20) by another set of relations, independent from (3.18): for $0 < s \leq s' \leq k$, one has

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\left(X_{s\alpha_i}(z)X_{s'\alpha_i}(z)\right) &= \frac{s+s'}{s}\left(\frac{\mathrm{d}}{\mathrm{d}z}X_{s\alpha_i}(z)\right)X_{s'\alpha_i}(z) \\ &= \frac{s+s'}{s-1}\left(\frac{\mathrm{d}}{\mathrm{d}z}X_{(s-1)\alpha_i}(z)\right)X_{(s'+1)\alpha_i}(z) \\ &= \cdots = \frac{s+s'}{1}\left(\frac{\mathrm{d}}{\mathrm{d}z}X_{\alpha_i}(z)\right)X_{(s'+s-1)\alpha_i}(z) \end{aligned} \tag{3.22}$$

(for $0 < s < s' \leq k$ these are s new independent relations). The vector counterparts of these relations are

$$\begin{aligned}
&x_{s\alpha_i}(-s-1)x_{s'\alpha_i}(-s')\cdot v(k\hat{\Lambda}_0)+x_{s\alpha_i}(-s)x_{s'\alpha_i}(-s'-1)\cdot v(k\hat{\Lambda}_0)\\
&=\frac{s+s'}{s}x_{s\alpha_i}(-s-1)x_{s'\alpha_i}(-s')\cdot v(k\hat{\Lambda}_0)\\
&=\frac{s+s'}{s-1}x_{(s-1)\alpha_i}(-(s-1)-1)x_{(s'+1)\alpha_i}(-(s'+1))\cdot v(k\hat{\Lambda}_0)\\
&=\cdots=\frac{s+s'}{1}x_{\alpha_i}(-1)x_{(s'+s-1)\alpha_i}(-(s'+s-1))\cdot v(k\hat{\Lambda}_0).
\end{aligned} \tag{3.23}$$

4. Quasi-particle basis for the principal subspaces of basic modules

Throughout this section we shall simply talk about quasi-particles without specifying their charge since they all will have charge 1.

Consider all the $n+1$ basic modules $L(\hat{\Lambda}_j)$, $0 \leq j \leq n$, as constructed in Section 2, i.e., set

$$X_\beta(z) := Y(e^\beta, z), \quad \beta \in \Delta, \tag{4.1}$$

(cf. (2.1)–(2.3)).

For every j, $0 \leq j \leq n$, we propose a basis for the corresponding principal subspace $W(\hat{\Lambda}_j) \subset L(\hat{\Lambda}_j)$. Here is the set of quasi-particle monomials which generate our basis (when acting on the highest weight vector $v(\hat{\Lambda}_j)$):

Definition 4.1. Fix j, $0 \leq j \leq n$. Set

$$\mathfrak{B}_{W(\hat{\Lambda}_j)} := \bigsqcup_{r_n,\ldots,r_1 \geq 0} \tag{4.2}$$

$$\left\{ x_{\alpha_n}(m_{r_n,n})\cdots x_{\alpha_n}(m_{1,n})\cdots\cdots x_{\alpha_1}(m_{r_1,1})\cdots x_{\alpha_1}(m_{1,1}) \right.$$

$$\left. \left| \begin{array}{l} m_{p,i} \in \mathbb{Z},\ 1 \leq i \leq n,\ 1 \leq p \leq r_i; \\ m_{p,i} \leq r_{i-1} - \delta_{i,j} - 2(p-1) - 1; \\ m_{p+1,i} \leq m_{p,i} - 2 \end{array} \right. \right\},$$

where $r_0 := 0$ (when $r_i = 0$, $1 \leq i \leq n$, no quasi-particles of color i are present).

The first of the two nontrivial conditions condition is a truncation condition which in particular incorporates the interaction between quasi-particles of color i and quasi-particles of color $i-1$ (quasi-particles of colors i and $i-p$, $p > 1$, do not interact with each other). The second condition is nothing else but the "difference two at distance one" condition for the quasi-particles of the same color i (cf. Preliminaries).

Example 4.1. Consider $\mathfrak{g} = sl(3)$ (i.e., $n = 2$) and the vacuum principal subspace $W(\hat{\Lambda}_0)$ (i.e., $j = 0$). We shall denote for brevity the monomial $x_{\alpha_2}(s)\cdots x_{\alpha_1}(t)$ by $(s_{\alpha_2}\ldots t_{\alpha_1})$. For the first few energy levels (the eigenvalues of the scaling operator D under the adjoint action), we list in Table 1 of the appendix the elements of $\mathfrak{B}_{W(\hat{\Lambda}_0)}$ of color-types $(1;2)$ and $(2;2)$.

Note that due to the second condition in the above definition, we can weaken the first condition and still get the same set:

$$\mathfrak{B}_{W(\hat{\Lambda}_j)} = \bigsqcup_{r_n,\ldots,r_1 \geq 0} \left\{ x_{\alpha_n}(m_{r_n,n})\cdots x_{\alpha_n}(m_{1,n})\cdots\cdots x_{\alpha_1}(m_{r_1,1})\cdots x_{\alpha_1}(m_{1,1}) \right. \tag{4.3}$$

$$\left. \begin{array}{|l} m_{p,i} \in \mathbb{Z},\ 1 \leq i \leq n,\ 1 \leq p \leq r_i; \\ m_{p,i} \leq r_{i-1} - \delta_{i,j} - 1; \\ m_{p+1,i} \leq m_{p,i} - 2 \end{array} \right\}.$$

The first nontrivial condition in (4.3) is encoded in the assertion of the following lemma.

Lemma 4.1. *Fix j, $0 \leq j \leq n$. One has*

$$\left[\prod_{i=2}^{n}\prod_{p=1}^{r_i}\prod_{q=1}^{r_{i-1}}\left(1 - \frac{z_{q,i-1}}{z_{p,i}}\right)\right] X_{\alpha_n}(z_{r_n,n})\cdots X_{\alpha_1}(z_{1,1})\cdot v(\hat{\Lambda}_j) \tag{4.4}$$

$$\in \left[\prod_{i=1}^{n}\prod_{p=1}^{r_i} z_{p,i}^{\delta_{i,j}-r_{i-1}}\right] W(\hat{\Lambda}_j)[[z_{r_n,n},\ldots,z_{1,1}]],$$

where $r_0 := 0$.

Proof. Follows from a more general claim:

$$\left[\prod_{\substack{i,l=1\\ i\geq l}}^{n}\prod_{p=1}^{r_i}\prod_{\substack{q=1\\ p>q \text{ for } i=l}}^{r_l} (z_{p,i} - z_{q,l})^{-\langle\alpha_i,\alpha_l\rangle}\right] X_{\alpha_n}(z_{r_n,n})\cdots X_{\alpha_1}(z_{1,1})\cdot v(\hat{\Lambda}_j) \tag{4.5}$$

$$\in \left[\prod_{i=1}^{n}\prod_{p=1}^{r_i} z_{p,i}^{\delta_{i,j}}\right] W(\hat{\Lambda}_j)[[z_{r_n,n},\ldots,z_{1,1}]],$$

where the binomial expressions are to be expanded as usual in nonnegative integral powers of the second variable. But this claim is an immediate implication of the fact that $\hat{\mathfrak{h}}^{+}\cdot v(\hat{\Lambda}_j) = 0$ and the commutation relation (2.8), applied to $\lambda = \alpha_l$, $\mu = \alpha_i$, $z_2 = z_{q,l}$, $z_1 = z_{p,i}$ (see also (2.5) and (2.6)). □

One should not fail to observe that on the left-hand side of (4.5) we have simply removed "by hand" all the "universal" poles and zeroes in the generating function

$$X_{\alpha_n}(z_{r_n,n})\cdots X_{\alpha_1}(z_{1,1})\cdot v(\hat{\Lambda}_j).$$

Namely, these are the (order one) poles on the hyperplanes

$$z_{p,i} = z_{q,i-1}, \quad 2 \le i \le n,\ 1 \le p \le r_i,\ 1 \le q \le r_{i-1} \tag{4.6}$$

and the (order two) zeroes on the hyperplanes

$$z_{p,i} = z_{q,i}, \quad 1 \le i \le n,\ 1 \le q < p \le r_i. \tag{4.7}$$

These singularities ought to be "blamed" for the interaction between the quasi-particles. It is clear how to identify the restricted dual to $W(\hat{\Lambda}_j)$ space with an appropriate space of symmetric (with respect to each group of variables $z_{r_i,i},\dots,z_{1,i}$) polynomials and thus make a connection with the Feigin–Stoyanovsky construction [16].

We are now all set for a quick demonstration of the spanning property of the declared basis.

Theorem 4.1. *For a given j, $0 \le j \le n$, the set $\{b\cdot v(\hat{\Lambda}_j) \mid b \in \mathfrak{B}_{W(\hat{\Lambda}_j)}\}$ spans the principal subspace $W(\hat{\Lambda}_j)$ of $L(\hat{\Lambda}_j)$.*

Proof. In view of Lemma 3.1, it suffices to show that every vector $b\cdot v(\hat{\Lambda}_j)$, b a monomial in U, is a linear combination of the proposed vectors.

If a monomial b of color-type $(r_n;\dots;r_1)$ violates the condition

$$m_{p,i} \le r_{i-1} - \delta_{i,j} - 1 \tag{4.8}$$

for some i, $1 \le i \le n$, and p, $1 \le p \le r_i$, then Lemma 4.1 implies that the vector $b\cdot v(\hat{\Lambda}_j)$ is a linear combination of vectors of the form $b'\cdot v(\hat{\Lambda}_j)$, b' a monomial from U, with b' and b having the same color-type and *total* index-sum and $b' \succ b$ (but there is at least one color i for which the corresponding index-sums in b' and b are different; cf. Section 3 where the ordering "$\prec$" for quasi-particle monomials was introduced). There are only finitely many such monomials b' which do not annihilate $v(\hat{\Lambda}_j)$.

On the other hand, (3.17) with $k = 1$ furnishes a new independent constraint for every pair of quasi-particles of the same color. It implies that if a monomial b from U violates the condition

$$m_{p+1,i} \le m_{p,i} - 2 \tag{4.9}$$

(the "difference two at distance one " condition for a given color i, $1 \le i \le n$), we can express the vector $b\cdot v(\hat{\Lambda}_j)$ as a linear combination of vectors of the form $b'\cdot v(\hat{\Lambda}_j)$, b' a monomial from U, with b' and b having the same color-type and the same index-sum for every given color i and in addition, $b' \succ b$ (note again that there are only finitely many such monomials b' which do not annihilate $v(\hat{\Lambda}_j)$).

We have shown that if a monomial violates either of the two nontrivial conditions in definition (4.3), it can be raised in the ordering "$\prec$" (which has finite intervals). In other words, after finitely many steps, we can express every vector $b \cdot v(\hat{\Lambda}_j)$, b a monomial from U, through vectors from the proposed set $\{b \cdot v(\hat{\Lambda}_j) \,|\, b \in \mathfrak{B}_{W(\hat{\Lambda}_j)}\}$. $\square$

Remark 4.1. We have actually proven that every vector $b \cdot v(\hat{\Lambda}_j)$, b a quasi-particle monomial from U, $b \notin \mathfrak{B}_{W(\hat{\Lambda}_j)}$, is a linear combination of vectors of the form $b' \cdot v(\hat{\Lambda}_j)$, $b' \in \mathfrak{B}_{W(\hat{\Lambda}_j)}$, with b' and b having the same color-type and total index-sum and moreover $b' \succ b$.

As suggested in Section 2, we are going to prove the independence of the above spanning set using intertwining operators between different modules (although there are other possible approaches, for example, through the dual picture of Feigin and Stoyanovsky [16]).

Theorem 4.2. *For a given j, $0 \le j \le n$, the set $\{b \cdot v(\hat{\Lambda}_j) | b \in \mathfrak{B}_{W(\hat{\Lambda}_j)}\}$ forms a basis for the principal subspace $W(\hat{\Lambda}_j)$ of $L(\hat{\Lambda}_j)$.*

Proof. We shall prove first that a monomial relation $b \cdot v(\hat{\Lambda}_j) = 0$, $b \in \mathfrak{B}_{W(\hat{\Lambda}_j)}$, would imply $v(\hat{\Lambda}_j) = 0$ and therefore is impossible. This will only show that each proposed basis vector is nonzero but a slight elaboration of this argument will later eliminate the possibility for *any* linear relation among the vectors and thus prove their independence.

Choose for concreteness a monomial of color-type $(r_n; \ldots; r_2; r_1)$

$$b := x_{\alpha_n}(m_{r_n,n}) \cdots x_{\alpha_2}(m_{1,2}) x_{\alpha_1}(m_{r_1,1}) \cdots x_{\alpha_1}(m_{1,1}) \in \mathfrak{B}_{W(\hat{\Lambda}_j)} \tag{4.10}$$

and assume that a relation $b \cdot v(\hat{\Lambda}_j) = 0$ holds. Apply

$$\operatorname{Res}_z \left(z^{-1-\langle \Lambda_1, \Lambda_j \rangle} \mathcal{Y}(e^{\Lambda_1}, z) \right) \tag{4.11}$$

on both sides of the relation and employ (2.11) to move this operator all the way to the right. Then use

$$\operatorname{Res}_z \left(z^{-1-\langle \Lambda_1, \Lambda_j \rangle} \mathcal{Y}(e^{\Lambda_1}, z) \right) \cdot v(\hat{\Lambda}_j) = \text{const}\, e^{\Lambda_1} \cdot v(\hat{\Lambda}_j), \quad \text{const} \in \mathbb{C}^{\times}, \tag{4.12}$$

and (2.4) to move back the operator e^{Λ_1} all the way to the left at the expense of increasing by one the indices of all the quasi-particles of color 1. Drop the invertible operator e^{Λ_1} and conclude that $b' \cdot v(\hat{\Lambda}_j) = 0$, where

$$b' = x_{\alpha_n}(m_{r_n,n}) \cdots x_{\alpha_2}(m_{1,2}) x_{\alpha_1}(m_{r_1,1} + 1) \cdots x_{\alpha_1}(m_{1,1} + 1) \in \mathfrak{B}_{W(\hat{\Lambda}_j)}. \tag{4.13}$$

Repeat the same trick until the rightmost index reaches its maximal allowed value (before the corresponding quasi-particle is annihilated by the highest weight vector), namely, $m_{1,1} = -1 - \delta_{1,j}$. Since

$$x_{\alpha_1}(-1 - \delta_{1,j}) \cdot v(\hat{\Lambda}_j) = \text{const}\, e^{\alpha_1} \cdot v(\hat{\Lambda}_j), \quad \text{const} \in \mathbb{C}^{\times}, \tag{4.14}$$

formula (2.4) allows to move the operator e^{α_1} all the way to the left at the expense of increasing by two the indices of all the quasi-particles of color 1 and decreasing by one the indices of all the quasi-particles of color 2. Dropping the invertible operator e^{α_1}, conclude that $b'' \cdot v(\hat{\Lambda}_j) = 0$, where

$$b'' = x_{\alpha_n}(m_{r_n,n}) \cdots x_{\alpha_2}(m_{1,2} - 1) x_{\alpha_1}(m_{r_1,1} + 2) \cdots x_{\alpha_1}(m_{2,1} + 2) \in \mathfrak{B}_{W(\hat{\Lambda}_j)} \tag{4.15}$$

is of color-type $(r_n; \ldots; r_2; r_1 - 1)$. Repeat this whole cycle and decrease the number of quasi-particles until none of them is left, i.e., the false identity $v(\hat{\Lambda}_j) = 0$ is obtained – contradiction!

Now assume that a general linear relation $\sum_{s=0}^{m} \xi_s b_s \cdot v(\hat{\Lambda}_j) = 0$ holds, $\xi_s \in \mathbb{C}^\times$, $b_s \in \mathfrak{B}_{W(\hat{\Lambda}_j)}$ all distinct (we can assume that the constraint is not a sum of two nontrivial constraints and this implies that all b_s are of the same color-type). Execute the above reduction procedure for the monomial which is smallest (among those involved) in the linear lexicographic ordering "$<$" (cf. (1.11) and Section 3) and enjoy the observation that – by the very definition of "$<$" – all the other monomials get annihilated at some intermediate stage of the reduction (because they are reduced to the form $\cdots x_{\alpha_i}(m)$ for some $m > -1 - \delta_{i,j}$, $1 \le i \le n$). So, the outcome is again the false identity $v(\hat{\Lambda}_j) = 0$ – contradiction! $\square$

It is truly remarkable that the above recursive reduction actually fails if we start with a monomial b from U which is not in $\mathfrak{B}_{W(\hat{\Lambda}_j)}$ (in other words we get at some stage of the induction to the trivial but true identity $0 = 0$). This means that we could have discovered our basis just from the requirement that the above *reductio ad absurdum* works!

Without any further elaborations, we can write down character formulas for the principal subspaces. The only two observations needed are the fundamental combinatorial identity (cf. [1])

$$\frac{1}{(q)_r} := \frac{1}{(1-q)(1-q^2)\cdots(1-q^r)} = \sum_{m \ge 0} \left\{ \begin{matrix} \text{number of partitions of } m \text{ with} \\ \text{at most } r \text{ parts} \end{matrix} \right\} q^m, \tag{4.16}$$

where $r \in \mathbb{Z}_+$, $(q)_0 := 1$, and the simple numerical identity

$$\sum_{p=1}^{r} (2(p-1)+1) = 1 + 3 + 5 + \cdots + (2r-1) = r^2. \tag{4.17}$$

From the very Definition 4.1, we immediately conclude that for every j, $0 \le j \le n$, one has

$$\operatorname{Tr} q^D \Big|_{W(\hat{\Lambda}_j)} = \sum_{r_1 \ge 0} \frac{q^{r_1^2 + r_1 \delta_{1,j}}}{(q)_{r_1}} \sum_{r_2 \ge 0} \frac{q^{r_2^2 + r_2(\delta_{2,j} - r_1)}}{(q)_{r_2}} \cdots \sum_{r_n \ge 0} \frac{q^{r_n^2 + r_n(\delta_{n,j} - r_{n-1})}}{(q)_{r_n}}. \tag{4.18}$$

Equivalently,

$$\operatorname{Tr} q^D \Big|_{W(\hat{\Lambda}_j)} = \sum_{r_1,\dots,r_n \geq 0} \frac{q^{\frac{1}{2}\sum_{l,m=1}^{n} A_{lm} r_l r_m}}{\prod_{i=1}^{n} (q)_{r_i}} q^{r_j}, \tag{4.19}$$

where $r_0 := 0$, and $(A_{lm})_{l,m=1}^{n}$ is the Cartan matrix of $\mathfrak{g} = sl(n+1, \mathbb{C})$.

In the case of the vacuum module ($j = 0$), this is the Feigin–Stoyanovsky character formula [16].

In terms of a generating function which encodes the quasi-particle structure of the basis, one has

$$\operatorname{ch} W(\hat{\Lambda}_j) = \sum_{r_1,\dots,r_n \geq 0} \frac{q^{1/2\sum_{l,m=1}^{n} A_{lm} r_l r_m}}{\prod_{i=1}^{n} (q)_{r_i}} q^{r_j} \prod_{i=1}^{n} y_i^{r_i}. \tag{4.20}$$

The coefficient of $y_1^{r_1} \cdots y_n^{r_n}$ on the right-hand side gives the q-character of the weight subspace $W_{\Lambda_j + \sum_{i=1}^{n} r_i \alpha_i}(\hat{\Lambda}_j)$.

5. Quasi-particle basis for the principal subspaces at any positive integral level k

Fix a level $k \in \mathbb{Z}_+$. We shall consider for simplicity only highest weights of the form

$$\hat{\Lambda} := k_0 \hat{\Lambda}_0 + k_j \hat{\Lambda}_j = k \hat{\Lambda}_0 + \Lambda, \quad \text{where } \Lambda := k_j \Lambda_j, \tag{5.1}$$

for some j, $1 \leq j \leq n$; $k_0, k_j \in \mathbb{N}$ and $k_0 + k_j = k$. (All dominant integral weights are of this form if $n = 1$, i.e., $\mathfrak{g} = sl(2, \mathbb{C})$ or if $k = 1$.) We would like to warn the reader that this restriction is not as innocuous as it might look: neither Definition 5.1 of our basis, nor the subsequent statements (e.g., the crucial for the spanning argument Lemma 5.1) are immediately generalizable for other highest weights.

It is convenient to define

$$j_t := \begin{cases} 0 & \text{for } 0 < t \leq k_0, \\ j & \text{for } k_0 < t \leq k = k_0 + k_j. \end{cases} \tag{5.2}$$

Simulating the level one bases built in the previous section, we shall propose a basis for the principal subspace $W(\hat{\Lambda}) \subset L(\hat{\Lambda})$, which will be generated by quasi-particles (of charge no greater than k) acting on the highest weight vector $v(\hat{\Lambda})$. Not surprisingly, our main technical tool will be the realization of $W(\hat{\Lambda})$ as a subspace of the tensor product of k level one modules. More precisely,

$$W(\hat{\Lambda}) = U(\bar{\mathfrak{n}}_+) \cdot v(\hat{\Lambda}) \subset W(\hat{\Lambda}_{j_k}) \otimes \cdots \otimes W(\hat{\Lambda}_{j_1}) \subset V_P^{\otimes k}, \tag{5.3}$$

where

$$\begin{aligned} v(\hat{\Lambda}) &:= v(\hat{\Lambda}_{j_k}) \otimes \cdots \otimes v(\hat{\Lambda}_{j_1}) \\ &= \underbrace{v(\hat{\Lambda}_j) \otimes \cdots \otimes v(\hat{\Lambda}_j)}_{k_j \textit{ factors}} \otimes \underbrace{v(\hat{\Lambda}_0) \otimes \cdots \otimes v(\hat{\Lambda}_0)}_{k_0 \textit{ factors}} \end{aligned} \tag{5.4}$$

and $U(\bar{\mathfrak{n}}_+)$ acts through Δ^{k-1} (the $(k-1)$-fold iterate of the standard coproduct Δ in the bialgebra $U(\bar{\mathfrak{n}}_+)$; $\Delta^0 := \mathrm{id}$); no confusion can arise from the fact that Δ denotes also the set of $\mathfrak{g}$-roots). In other words, we set

$$\begin{aligned} X_\beta(z) &:= \Delta^{k-1}(Y(e^\beta, z)) \\ &= \underbrace{Y(e^\beta,z) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{k \textit{ factors}} + \underbrace{\mathbf{1} \otimes Y(e^\beta,z) \otimes \cdots \otimes \mathbf{1}}_{k \textit{ factors}} + \cdots \\ &\quad + \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes Y(e^\beta,z)}_{k \textit{ factors}}, \quad \beta \in \Delta. \end{aligned} \tag{5.5}$$

Note that by Lemma 3.1, one has $W(\hat{\Lambda}) = U \cdot v(\hat{\Lambda})$.

We shall freely use all the notations, definitions and formulas from Section 3.

Below is the set of quasi-particle monomials from U which generate our basis (see the subsequent example). We keep the format of Definition 4.1 in order to help the reader to get through this confusing abundance of indices (we shall alleviate the pain by pinpointing the correspondence between the entries in Definition 4.1 and their generalizations in the current context). Our set is a disjoint union over color-charge–types (cf. Section 3)

$$(n_{r_n^{(1)},n}, \ldots, n_{1,n}; \ldots; n_{r_1^{(1)},1}, \ldots, n_{1,1}),$$

$$0 < n_{r_i^{(1)},i} \le \cdots \le n_{2,i} \le n_{1,i} \le k, \ \sum_{p=1}^{r_i^{(1)}} n_{p,i} =: r_i, \ 1 \le i \le n, \tag{5.6}$$

or, equivalently, over the color-dual-charge-types

$$(r_n^{(1)}, \ldots, r_n^{(k)}; \ldots; r_1^{(1)}, \ldots, r_1^{(k)}),$$

$$r_i^{(1)} \ge r_i^{(2)} \ge \cdots \ge r_i^{(k)} \ge 0, \ \sum_{t=1}^{k} r_i^{(t)} = r_i, \ 1 \le i \le n \tag{5.7}$$

($r_i^{(1)}$ distinct quasi-particles of color i and charge at most k are present).

Definition 5.1. Fix a highest weight $\hat{\Lambda}$ as in (5.1). Set

$$\mathfrak{B}_{W(\hat{\Lambda})} := \bigsqcup_{\substack{0 \le n_{r_n^{(1)},n} \le \cdots \le n_{1,n} \le k \\ \cdots \\ 0 \le n_{r_1^{(1)},1} \le \cdots \le n_{1,1} \le k}} \left(\text{or, equivalently,} \bigsqcup_{\substack{r_n^{(1)} \ge \cdots \ge r_n^{(k)} \ge 0 \\ \cdots \\ r_1^{(1)} \ge \cdots \ge r_1^{(k)} \ge 0}} \right) \tag{5.8}$$

$$\left\{ x_{n_{r_n^{(1)},n}\alpha_n}(m_{r_n^{(1)},n}) \cdots x_{n_{1,n}\alpha_n}(m_{1,n}) \cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(m_{1,1}) \;\middle|\; \begin{array}{l} m_{p,i} \in \mathbb{Z},\ 1 \le i \le n,\ 1 \le p \le r_i^{(1)}; \\ m_{p,i} \le \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q,i-1}\} - \sum_{t=1}^{n_{p,i}} \delta_{i,j_t} \\ \qquad\qquad - \sum_{p>p'>0} 2\min\{n_{p,i}, n_{p',i}\} - n_{p,i}; \\ m_{p+1,i} \le m_{p,i} - 2n_{p,i} \text{ for } n_{p+1,i} = n_{p,i} \end{array} \right\},$$

where $r_0^{(1)} := 0$ and j_t was defined in (5.2).

It is rather obvious that the role of r_i (the total charge due to color i) in Definition 4.1 is played here by

$$\sum_{q=1}^{r_i^{(1)}} n_{q,i} = \sum_{t=1}^{k} r_i^{(t)} =: r_i$$

and moreover, a quasi-particle $x_{\alpha_i}(m_{p,i})$ of charge 1 from Definition 4.1 is simply replaced here by a quasi-particle $x_{n_{p,i}\alpha_i}(m_{p,i})$ of charge $n_{p,i}$. The "difference two at distance one" condition for quasi-particles of the same color and charge 1 in Definition 4.1 is generalized in the current setting to a "difference $2n_{p,i}$ at distance one" condition for quasi-particles of the same color and charge $n_{p,i}$. Not surprisingly, the quasi-particles of the same color, but of *different* charge, have to be treated as different objects since they do not satisfy any reasonable difference conditions among themselves (although they commute). The first (truncation) condition in Definition 4.1 has its entries (on the right-hand side of the inequality) generalized here as follows:

$$r_{i-1} = \sum_{q=1}^{r_{i-1}} 1 \mapsto \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q,i-1}\},$$

$$-\delta_{i,j} \mapsto -\sum_{t=1}^{n_{p,i}} \delta_{i,j_t},$$

$$-2(p-1) = -\sum_{p>p'>0} 2 \mapsto -\sum_{p>p'>0} 2\min\{n_{p,i}, n_{p',i}\} = -\sum_{p>q>0} 2n_{p,i},$$

$$-1 \mapsto -n_{p,i}.$$

Example 5.1. Consider $\mathfrak{g} = sl(3)$ (i.e., $n = 2$), $k = 2$ and the vacuum principal subspace $W(2\hat{\Lambda}_0)$. We shall denote for brevity the quasi-particle monomial $x_{s'\alpha_2}(s) \cdots x_{t'\alpha_1}(t)$ by $(s_{s'\alpha_2} \ldots t_{t'\alpha_1})$. For the first few energy levels (the eigenvalues of the scaling operator D under the adjoint action), we list in Table 2 of the appendix the elements of $\mathfrak{B}_{W(2\hat{\Lambda}_0)}$ of color-types $(1;2)$ and $(2;2)$.

It is illuminating to have the entries in this truncation condition written down in terms of the color-dual-charge-type parameters $r_i^{(t)}$. Suppose $n_{p,i} = s$, $1 \le s \le k$. Then

$$\sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q,i-1}\} = \sum_{q=1}^{r_{i-1}^{(1)}} \min\{s, n_{q,i-1}\} = \sum_{t=1}^{s} r_{i-1}^{(t)}. \tag{5.9}$$

Since the number of quasi-particles of charge s and color i is $r_i^{(s)} - r_i^{(s+1)}$, the total "shift" due to the interaction between quasi-particles of colors i and $i-1$ is

$$\sum_{s=1}^{k}(r_i^{(s)} - r_i^{(s+1)})\sum_{t=1}^{s} r_{i-1}^{(t)} = r_i^{(1)}r_{i-1}^{(1)} + r_i^{(2)}r_{i-1}^{(2)} + \cdots + r_i^{(k)}r_{i-1}^{(k)} \tag{5.10}$$

(this and the subsequent identities will be needed later for the character formulas which are written in terms of $r_i^{(s)}$). Similarly, the total "shift" due to the delta functions is

$$-\sum_{t=1}^{k} r_i^{(t)}\delta_{i,j_t}. \tag{5.11}$$

A longer but straightforward calculation shows that the total "shift" due to the interaction between quasi-particles of the same color i is

$$-\sum_{p=1}^{r_i^{(1)}}\left(\sum_{p>p'>0} 2\min\{n_{p,i}, n_{p',i}\} + n_{p,i}\right) = -\sum_{\substack{p,p'=1 \\ p>p'}}^{r_i^{(1)}} 2n_{p,i} - \sum_{t=1}^{k} r_i^{(t)} = -\sum_{t=1}^{k} r_i^{(t)^2} \tag{5.12}$$

(cf. [16, Theorem 2.7.1]).

Similarly to the transition between Definition 4.1 and the equivalent definition (4.3), we can weaken the first condition in Definition 5.1, omitting the terms on the right-hand side which are due to the interaction between quasi-particles of the same color and charge (the second condition implies that the original inequalities automatically hold). Since for $n_{p,i} = s$, one has

$$-\sum_{n_{p',i} > n_{p,i}} 2\min\{n_{p,i}, n_{p',i}\} = -\sum_{n_{p',i} > n_{p,i}} 2n_{p,i} = -2s\sum_{t=s+1}^{k} r_i^{(t)}, \tag{5.13}$$

the set $\mathfrak{B}_{W(\hat{\Lambda})}$ can be alternatively described as follows (cf. also (5.9)):

$$\mathfrak{B}_{W(\hat{\Lambda})} := \bigsqcup_{\substack{0 \le n_{r_n^{(1)},n} \le \cdots \le n_{1,n} \le k \\ \cdots \\ 0 \le n_{r_1^{(1)},1} \le \cdots \le n_{1,1} \le k}} \left(\text{or, equivalently,} \quad \bigsqcup_{\substack{r_n^{(1)} \ge \cdots \ge r_n^{(k)} \ge 0 \\ \cdots \\ r_1^{(1)} \ge \cdots \ge r_1^{(k)} \ge 0}}\right) \tag{5.14}$$

$$\left\{ x_{n_{r_n^{(1)},n}\alpha_n}(m_{r_n^{(1)},n})\cdots x_{n_{1,n}\alpha_n}(m_{1,n})\cdots\cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1}) \,\middle|\, \begin{array}{ll} m_{p,i} & \in \mathbb{Z},\ 1 \le i \le n,\ 1 \le p \le r_i^{(1)}; \\ m_{p,i} & \le \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q,i-1}\} - \sum_{t=1}^{n_{p,i}} \delta_{i,j_t} - \sum_{n_{p',i} > n_{p,i}} 2n_{p,i} - n_{p,i} \\ & = \sum_{t=1}^{s} r_{i-1}^{(t)} - \sum_{t=1}^{s} \delta_{i,j_t} - 2s \sum_{t=s+1}^{k} r_i^{(t)} - s, \text{ where } s := n_{p,i}; \\ m_{p+1,i} & \le m_{p,i} - 2n_{p,i} \text{ for } n_{p+1,i} = n_{p,i} \end{array} \right\}.$$

In contrast to the level one basis (4.3), we have here a new term

$$-\sum_{n_{p',i} > n_{p,i}} 2n_{p,i} = -2s \sum_{t=s+1}^{k} r_i^{(t)} \tag{5.15}$$

(with no analog in (4.3)), incorporating the interaction between a given quasi-particle of charge $n_{p,i} = s$ and all the quasi-particles of the same color but greater charges.

The following lemma is the higher level generalization of Lemma 4.1.

Lemma 5.1. *Fix a highest weight $\hat{\Lambda}$ as in (5.1). For a generating function*

$$X_{n_{r_n^{(1)},n}\alpha_n}(z_{r_n^{(1)},n})\cdots X_{n_{1,1}\alpha_1}(z_{1,1})$$

of color-charge-type

$$(n_{r_n^{(1)},n}, \ldots, n_{1,n}; \ldots; n_{r_1^{(1)},1}, \ldots, n_{1,1})$$

(cf. (5.6)) and a corresponding color-dual-charge-type

$$(r_n^{(1)}, \ldots, r_n^{(k)}; \ldots; r_1^{(1)}, \ldots, r_1^{(k)}),$$

(cf.(5.7)), one has

$$\left[\prod_{i=2}^{n} \prod_{p=1}^{r_i^{(1)}} \prod_{q=1}^{r_{i-1}^{(1)}} \left(1 - \frac{z_{q,i-1}}{z_{p,i}}\right)^{\min\{n_{p,i}, n_{q,i-1}\}}\right] X_{n_{r_n^{(1)},n}\alpha_n}(z_{r_n^{(1)},n})\cdots X_{n_{1,1}\alpha_1}(z_{1,1}) \cdot v(\hat{\Lambda})$$

$$\in \left[\prod_{i=1}^{n} \prod_{p=1}^{r_i^{(1)}} z_{p,i}^{\sum_{t=1}^{n_{p,i}} \delta_{i,j_t} - \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q,i-1}\}}\right] W(\hat{\Lambda})[[z_{r_n^{(1)},n}, \ldots, z_{1,1}]], \tag{5.16}$$

where $r_0^{(1)} := 0$.

Proof. Follows immediately from Lemma 4.1, (3.17) with $k = 1$ and the explicit form (5.5) of the iterated coproduct Δ^{k-1}. Note that we are able to encode the highest weight $\hat{\Lambda}$ in the simple term $\sum_{t=1}^{n_{p,i}} \delta_{i,j_t}$ only because of the special choice (5.1). □

We are now ready to state the higher level generalization of the spanning Theorem 4.1. We continue our strategy of imitating the level one picture. The only essentially new element in the proof below (as compared to the proof of Theorem 4.1) is the separate treating of the new term $-\sum_{n_{p',i}>n_{p,i}} 2n_{p,i}$, incorporating the interaction between quasi-particles of the same color but different charges (cf. (5.14) and (5.15)).

Theorem 5.1. *For a given highest weight $\hat{\Lambda}$ as in (5.1), the set $\{b\cdot v(\hat{\Lambda}) \mid b \in \mathfrak{B}_{W(\hat{\Lambda})}\}$ spans the principal subspace $W(\hat{\Lambda})$ of $L(\hat{\Lambda})$.*

Proof. Since Lemma 3.1 holds for any level, it suffices to show that every vector $b\cdot v(\hat{\Lambda})$, b a quasi-particle monomial from U, is a linear combination of the proposed vectors.

Suppose a quasi-particle monomial b of color-charge-type (5.6) (and a corresponding dual-color-charge-type (5.7)) violates the condition

$$m_{p,i} \leq \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q,i-1}\} - \sum_{t=1}^{n_{p,i}} \delta_{i,j_t} - n_{p,i} = \sum_{t=1}^{s} r_{i-1}^{(t)} - \sum_{t=1}^{s} \delta_{i,j_t} - s, \qquad (5.17)$$

where $s := n_{p,i}$, and $1 \leq i \leq n$, $1 \leq p \leq r_i^{(1)}$ (this is the first nontrivial condition in (5.14) with the term $-\sum_{n_{p',i}>n_{p,i}} 2n_{p,i}$ dropped). Then Lemma 5.1 implies that the vector $b\cdot v(\hat{\Lambda})$ is a linear combination of vectors of the form $b'\cdot v(\hat{\Lambda})$, b' a quasi-particle monomial from U, $b' \succ b$, with b' and b having the same color-charge-type and *total* index-sum (but there is at least one color i for which the corresponding index-sums in b' and b are different). There are only finitely many such quasi-particle monomials b' which do not annihilate $v(\hat{\Lambda})$.

Now the constraints (3.18) and (3.22) come to action. They furnish $2s$ new independent (nontrivial) relations for monochromatic quasi-particle monomials of (color)-charge-type (s, s'), $0 < s < s' \leq k$. Some of these relations involve quasi-particle monomials of the same (color)-type, but of greater (color)-charge-types. Adding these new relations, we can strengthen the inequality from the last paragraph and claim that if a quasi-particle monomial b of color-charge-type (5.6) violates the stronger condition

$$\begin{aligned} m_{p,i} &\leq \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q,i-1}\} - \sum_{t=1}^{n_{p,i}} \delta_{i,j_t} - \sum_{n_{p',i}>n_{p,i}} 2n_{p,i} - n_{p,i} \\ &= \sum_{t=1}^{s} r_{i-1}^{(t)} - \sum_{t=1}^{s} \delta_{i,j_t} - 2s \sum_{t=s+1}^{k} r_i^{(t)} - s, \end{aligned} \qquad (5.18)$$

where $s := n_{p,i}$, and $1 \leq i \leq n$, $1 \leq p \leq r_i^{(1)}$ (this is exactly the first nontrivial condition in (5.14)), then the vector $b \cdot v(\hat{\Lambda})$ is still a linear combination of vectors

$b' \cdot v(\hat{\Lambda})$, b' a quasi-particle monomial from U, $b' \succ b$, with b' and b having the same color-type and *total* index-sum (but the color-charge-type is typically different; no need to say that there are only finitely many such quasi-particle monomials b' which do not annihilate $v(\hat{\Lambda})$). In order to see this, one has to be more patient than usual and induct on the number $\sum_{t=s+1}^{k} r_i^{(t)}$ of quasi-particles of color i and charge greater than $s = n_{p,i}$ (this is exactly the number of summands in the new term (5.15) which distinguishes (5.18) from its predecessor (5.17)). Namely, using the $2n_{p,i}$ new constraints for our quasi-particle $x_{n_{p,i}\alpha_i}(m_{p,i})$ and its closest right neighbor of the same color i and greater charge, one obtains vectors of the desired form and (finitely many) other vectors of the form $b'' \cdot v(\hat{\Lambda})$, where b'' is of the same color-charge-type as b but its quasi-particle $x_{n_{p,i}\alpha_i}$ has index $\geq m_{p,i} + 2n_{p,i}$. It remains to use the inductive assumption for each of the vectors $b'' \cdot v(\hat{\Lambda})$ and observe that they are all expressible through vectors $b' \cdot v(\hat{\Lambda})$ of the desired form, i.e., such that not only $b' \succ b''$, but also $b' \succ b$!

Finally, the constraints (3.18) furnish s new independent (nontrivial) relations for monochromatic quasi-particle monomials of (color)-charge-type (s,s), $0 < s \leq k$. This means that if a quasi-particle monomial b of color-charge-type (5.6) violates the "difference $2n_{p,i}$ at distance one" condition

$$m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \quad \text{for} \quad n_{p+1,i} = n_{p,i} =: s, \tag{5.19}$$

(this is the second nontrivial condition in (5.14)), then the vector $b \cdot v(\hat{\Lambda})$ is a linear combination of vectors of the form $b' \cdot v(\hat{\Lambda})$, b' a quasi-particle monomial from U, $b' \succ b$, with b' and b having the same color-type and index-sum for any given color i (but the color-charge-type might be different; there are again only finitely many such quasi-particle monomials b' which do not annihilate $v(\hat{\Lambda})$).

Together with the conclusion of the previous paragraph and formula (5.14), this guarantees that – after finitely many steps – we can express every vector $b \cdot v(\hat{\Lambda})$, b a quasi-particle monomial from U, through vectors from the proposed set $\{b \cdot v(\hat{\Lambda}) \mid b \in \mathfrak{B}_{W(\hat{\Lambda})}\}$. $\square$

Remark 5.1. The proof implies that Remark 4.1 is true for level k highest weights: Every vector $b \cdot v(\hat{\Lambda})$, b a quasi-particle monomial from U, $b \notin \mathfrak{B}_{W(\hat{\Lambda})}$, is a linear combination of vectors of the form $b' \cdot v(\hat{\Lambda})$, $b' \in \mathfrak{B}_{W(\hat{\Lambda})}$, $b' \succ b$ with b' and b having the same color-type and total index-sum.

We proceed with the definition of a projection needed for generalizing the independence Theorem 4.2 to level k.

Consider the direct sum decomposition

$$W(\hat{\Lambda}_{j_k}) \otimes \cdots \otimes W(\hat{\Lambda}_{j_1}) = \coprod_{\substack{r_n^{(k)},\dots,r_1^{(k)} \geq 0 \\ \cdots \\ r_n^{(1)},\dots,r_1^{(1)} \geq 0}} W(\hat{\Lambda}_{j_k})_{(r_n^{(k)};\dots;r_1^{(k)})} \otimes \cdots \otimes W(\hat{\Lambda}_{j_1})_{(r_n^{(1)};\dots;r_1^{(1)})}, \tag{5.20}$$

(cf. (3.4) and (5.2)). For a chosen color-charge-type (5.6) and corresponding color-dual-charge-type

$$(r_n^{(1)},\dots,r_n^{(k)};\dots;r_1^{(1)},\dots,r_1^{(k)})$$

(cf. (5.7)), set

$$\pi_{(r_n^{(1)};\dots;r_1^{(k)})} : W(\hat{\Lambda}_{j_k})\otimes\cdots\otimes W(\hat{\Lambda}_{j_1}) \to W(\hat{\Lambda}_{j_k})_{(r_n^{(k)};\dots;r_1^{(k)})}\otimes\cdots\otimes W(\hat{\Lambda}_{j_1})_{(r_n^{(1)};\dots;r_1^{(1)})} \tag{5.21}$$

to be the projection given by the above decomposition (we shall denote by the same letter the obvious generalization of this projection to the space of formal series with coefficients in $W(\hat{\Lambda}_{j_k})\otimes\cdots\otimes W(\hat{\Lambda}_{j_1})$). For a generating function of the chosen color-charge-type, one can now conclude from the definition (5.5) and the constraint (3.17) (with $k=1$) that

$$\begin{aligned}
&\pi_{(r_n^{(1)};\dots;r_1^{(k)})}\cdot X_{n_{r_n^{(1)},n}\alpha_n}(z_{r_n^{(1)},n})\cdots X_{n_{1,1}\alpha_1}(z_{1,1})\cdot v(\hat{\Lambda}) \qquad (5.22)\\
&= \mathrm{const}\, Y(e^{\alpha_n},z_{r_n^{(k)},n})\cdots Y(e^{\alpha_n},z_{1,n})\cdots Y(e^{\alpha_1},z_{r_1^{(k)},1})\cdots Y(e^{\alpha_1},z_{1,1})\cdot v(\hat{\Lambda}_{j_k})\\
&\otimes\cdots\otimes\\
&\otimes Y(e^{\alpha_n},z_{r_n^{(1)},n})\cdots Y(e^{\alpha_n},z_{1,n})\cdots Y(e^{\alpha_1},z_{r_1^{(1)},1})\cdots Y(e^{\alpha_1},z_{1,1})\cdot v(\hat{\Lambda}_{j_1}),
\end{aligned}$$

where const $\in \mathbb{C}^\times$(a tensor product of k factors). In order to see this, fix a color i, $1\le i\le n$, and first "accommodate" the $n_{1,i}$ vertex operators $Y(e^{\alpha_i},z_{1,i})$ whose product generates the i-colored quasi-particles of charge $n_{1,i}$ (the greatest charge for color i in our monomial) – the projection π forces them to spread only along the $n_{1,i}$ rightmost tensor slots and in addition, (3.17) (with $k=1$) ensures that at most one (and hence, exactly one) vertex operator is apllied on each of these tensor slots. Proceed with the remaining vertex operators of color i in the very same fashion. Therefore, for a given quasi-particle monomial b of the above type, the projection $\pi_{(r_n^{(1)};\dots;r_1^{(k)})}\cdot b\cdot v(\hat{\Lambda})$ is a sum of tensor products of k charge-one-quasi-particle monomials (acting on $v(\hat{\Lambda})$), such that every quasi-particle of charge s from b has exactly one representative (level one quasi-particle of charge 1) on each of the s rightmost tensor slots and only there. To put it in different words, when a color i is fixed, one might associate the s^{th} tensor slot (counted from right to left) of the above tensor product with the s^{th} row (counted from the bottom to the top) of the Young diagram in Section 3, filling each box with a level one quasi-particle of charge one.

We can now take on the independence of the proposed basis vectors, generalizing the level one independence proof (Theorem 4.2). The projection π introduced above is needed to ensure that if a level one intertwining operators "shuttles" along the sth tensor slot (counted from right to left) as in the proof of Theorem 4.2, it can shift the indices of the quasi-particles of charge s (and the same color) without affecting quasi-particles of smaller charges. The reader is advised to look back at the proof of Theorem 4.2, since we shall only sketch here the modifications needed to carry out the argument in the present setting.

Theorem 5.2. *For a given highest weight $\hat{\Lambda}$ as in (5.1), the set $\{b \cdot v(\hat{\Lambda}) | b \in \mathfrak{B}_{W(\hat{\Lambda})}\}$ is indeed a basis for the principal subspace $W(\hat{\Lambda})$ of $L(\hat{\Lambda})$.*

Proof. Pick a quasi-particle monomial b from $\mathfrak{B}_{W(\hat{\Lambda})}$,

$$b := x_{n_{r_n^{(1)},n}\alpha_n}(m_{r_n^{(1)},n}) \cdots x_{n_{1,n}\alpha_n}(m_{1,n}) \cdots\cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(m_{1,1}) \tag{5.23}$$

of color-charge-type

$$(n_{r_n^{(1)},n}, \dots, n_{1,n}; \dots ; n_{r_1^{(1)},1}, \dots, n_{1,1})$$

(cf. (5.6)) and corresponding color-dual-charge-type

$$(r_n^{(1)}, \dots, r_n^{(k)}; \dots ; r_1^{(1)}, \dots, r_1^{(k)}) = (r_n^{(1)}, \dots, r_n^{(k)}; \dots ; r_1^{(1)}, \dots, r_1^{(s)}, 0, \dots, 0), \quad s := n_{1,1},$$

(cf. (5.7)). Assume that $b \cdot v(\hat{\Lambda}) = 0$ and hence, $\pi_{(r_n^{(1)};\dots;r_1^{(k)})} \cdot b \cdot v(\hat{\Lambda}) = 0$. Shuttling back and forth along the left-hand side of the last identity with the operator

$$\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \operatorname{Res}_z \left(z^{-1-\langle \Lambda_1, \Lambda_{j_s} \rangle} \mathcal{Y}(e^{\Lambda_1}, z)\right) \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{s-1 \text{ factors}}, \tag{5.24}$$

increase the indices of all the quasi-particles of charge $s = n_{1,1}$ and color 1 until the index of the rightmost such quasi-particle reaches its maximal allowed value (before this quasi-particle gets annihilated by the highest weight vector), namely, $m_{1,1} = -s - \sum_{t=1}^{s} \delta_{1,j_t}$ (cf. the proof of Theorem 4.2). Note that none of the quasi-particles in $\pi_{(r_n^{(1)};\dots;r_1^{(k)})} \cdot b \cdot v(\hat{\Lambda})$ of smaller charge or different color is affected by these operations and all the newly obtained vectors are still projections of quasi-particle monomials from $\mathfrak{B}_{W(\hat{\Lambda})}$ acting on $v(\hat{\Lambda})$. In other words, we end up with the vector

$$\pi_{(r_n^{(1)};\dots;r_1^{(k)})} \cdot b' x_{s\alpha_1}(-s - \sum_{t=1}^{s} \delta_{1,j_t}) \cdot v(\hat{\Lambda}) \tag{5.25}$$

$$= \operatorname{const} \pi_{(r_n^{(1)};\dots;r_1^{(k)})} \cdot b' \left(\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underbrace{e^{\alpha_1} \otimes \cdots \otimes e^{\alpha_1}}_{s \text{ factors}} \right) \cdot v(\hat{\Lambda})$$

for some const $\in \mathbb{C}^\times$ and a quasi-particle monomial $b' \in \mathfrak{B}_{W(\hat{\Lambda})}$ of color-charge-type

$$(n_{r_n^{(1)},n}, \dots, n_{1,n}; \dots ; n_{r_1^{(1)},1}, \dots, n_{2,1})$$

and corresponding color-dual-charge-type

$$(r_n^{(1)}, \dots, r_n^{(k)}; \dots ; r_1^{(1)} - 1, \dots, r_1^{(s)} - 1, 0 \dots, 0)$$

(i.e., the rightmost quasi-particle of color 1 and charge $n_{1,1}$ is not present in b''). Now formula (2.4) allows to move the invertible operator

$$\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underbrace{e^{\alpha_1} \otimes \cdots \otimes e^{\alpha_1}}_{s\,\text{factors}}$$

all the way to the left and drop it. The result is the relation $\pi_{(r_n^{(1)},\ldots,r_1^{(s)}-1,0,\ldots,0)} \cdot b'' \cdot v(\hat{\Lambda}) = 0$ for a quasi-particle monomial $b'' \in \mathfrak{B}_{W(\hat{\Lambda})}$ with one quasi-particle less (and of the same color-charge-type as b'). Keep decreasing the number of quasi-particles in the very same fashion until none of them is left, i.e., the false identity $v(\hat{\Lambda}) = 0$ is obtained – contradiction!

Finally, if a general linear relation holds, execute the above reduction for the quasi-particle monomial, minimal in the linear lexicographic ordering "$<$" (we can assume without loss of generality that all the quasi-particle monomials involved are of the same color-type and have the same total index-sum). During the process, the quasi-particle monomials with greater (in "$<$") color-charge-types are eliminated due to the projection π and the truncation (3.17) (with $k = 1$), while the quasi-particle monomials of the same color-charge-type, but with greater (in "$<$") index sequence, are annihilated by the vacuum vector just like in the level one picture. The result is again the false identity $v(\hat{\Lambda}) = 0$ – contradiction! □

As expected, the above reasoning works *only* for our very special quasi-particle monomials from $\mathfrak{B}_{W(\hat{\Lambda})}$. Indeed, it was our ambition to employ such an independence argument that served as a heuristic for discovering the basis-generating set $\mathfrak{B}_{W(\hat{\Lambda})}$!

Let us devote our final effort to writing down a character formula for $W(\hat{\Lambda})$ corresponding to the above basis. From the very Definition 5.1 and (4.16), (5.10), (5.11), (5.12), one has for a highest weight $\hat{\Lambda}$ as in (5.1) the following character formula:

$$\begin{aligned}
\operatorname{Tr} q^D \Big|_{W(\hat{\Lambda})} &= \sum_{r_1^{(1)} \geq \ldots \geq r_1^{(k)} \geq 0} \frac{q^{r_1^{(1)2}+\ldots+r_1^{(k)2}+\sum_{t=1}^{k} r_1^{(t)}\delta_{1,j_t}}}{(q)_{r_1^{(1)}-r_1^{(2)}} \cdots (q)_{r_1^{(k-1)}-r_1^{(k)}} (q)_{r_1^{(k)}}} \\
&\times \sum_{r_2^{(1)} \geq \ldots \geq r_2^{(k)} \geq 0} \frac{q^{r_2^{(1)2}+\cdots+r_2^{(k)2}-r_2^{(1)}r_1^{(1)}-\cdots-r_2^{(k)}r_1^{(k)}+\sum_{t=1}^{k} r_2^{(t)}\delta_{2,j_t}}}{(q)_{r_2^{(1)}-r_2^{(2)}} \cdots (q)_{r_2^{(k-1)}-r_2^{(k)}} (q)_{r_2^{(k)}}} \\
&\times \sum_{r_n^{(1)} \geq \ldots \geq r_n^{(k)} \geq 0} \frac{q^{r_n^{(1)2}+\ldots+r_n^{(k)2}-r_n^{(1)}r_{n-1}^{(1)}-\ldots-r_n^{(k)}r_{n-1}^{(k)}+\sum_{t=1}^{k} r_n^{(t)}\delta_{n,j_t}}}{(q)_{r_n^{(1)}-r_n^{(2)}} \cdots (q)_{r_n^{(k-1)}-r_n^{(k)}} (q)_{r_n^{(k)}}}
\end{aligned} \tag{5.26}$$

(cf.(5.2) where j_t is introduced).

Just like in the level one case, we can rewrite this formula in a more compact matrix form. Namely, set $p_i^{(s)} := r_i^{(s)} - r_i^{(s+1)}$, $1 \leq s \leq k-1$ and $p_i^{(k)} := r_i^{(k)}$ (note that $p_i^{(s)}$ is exactly the number of quasi-particles of color i and charge s). Since $\hat{\Lambda} = k_0\hat{\Lambda}_0 + k_j\hat{\Lambda}_j$ for some j, $1 \leq j \leq n$, we have $j_t = 0$ for $0 < t \leq k_0$ and $j_t = 1$ for $k_0 < t \leq k$.

A straightforward calculation now shows that the above expression can be rewritten as follows:

$$\operatorname{Tr} q^D \Big|_{W(k_0\hat{\Lambda}_0+k_j\hat{\Lambda}_j)} = \sum_{\substack{p_1^{(1)},\ldots,p_1^{(k)}\geq 0 \\ \cdots\cdots \\ p_n^{(1)},\ldots,p_n^{(k)}\geq 0,}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,k} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^n \prod_{s=1}^k (q)_{p_i^{(s)}}} q^{\tilde{p}_j} \tag{5.27}$$

where

$$\tilde{p}_j := p_j^{(k_0+1)} + 2p_j^{(k_0+2)} + \ldots + \; k_j p_j^{(k)},$$

$(A_{lm})_{l,m=1}^n$ is the Cartan matrix of $\mathfrak{g} = sl(n+1,\mathbb{C})$ and $B^{st} := \min\{s,t\}$, $1 \leq s,t \leq k$.

In the case of the vacuum module ($\hat{\Lambda} = k\hat{\Lambda}_0$), this is the Feigin–Stoyanovsky character formula announced in [16].

In terms of a generating function with formal variables $y_1^{(1)}, \ldots y_n^{(k)}$ (respectively, $y_1, \ldots, y_n$)which encode the color-charge-type (resp. the color-type) of the basis, one has

$$\begin{aligned} \operatorname{ch} W(k_0\hat{\Lambda}_0 + k_j\hat{\Lambda}_j) &= \sum_{\substack{p_1^{(1)},\ldots,p_1^{(k)}\geq 0 \\ \cdots\cdots \\ p_n^{(1)},\ldots,p_n^{(k)}\geq 0,}} \frac{q^{1/2\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,k} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^n \prod_{s=1}^k (q)_{p_i^{(s)}}} q^{\tilde{p}_j} \prod_{i=1}^n \prod_{s=1}^k (y_i^{(s)})^{p_i^{(s)}} \\ &= \sum_{\substack{p_1^{(1)},\ldots,p_1^{(k)}\geq 0 \\ \cdots\cdots \\ p_n^{(1)},\ldots,p_n^{(k)}\geq 0,}} \frac{q^{1/2\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,k} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^n \prod_{s=1}^k (q)_{p_i^{(s)}}} q^{\tilde{p}_j} \prod_{i=1}^n y_i^{\sum_{s=1}^k s p_i^{(s)}}. \end{aligned} \tag{5.28}$$

The coefficient of

$$(y_1^{(1)})^{p_1^{(1)}} \cdots (y_1^{(k)})^{p_1^{(k)}} \cdots (y_n^{(1)})^{p_n^{(1)}} \cdots (y_n^{(k)})^{p_n^{(k)}}$$

in the first expression gives the q-character of the subspace generated by quasi-particle monomials with exactly $p_i^{(s)}$ quasi-particles of color i and charge s (hence the color-charge-type of such monomials is

$$(\underbrace{1,\ldots,1}_{p_n^{(1)}},\ldots,\underbrace{k,\ldots,k}_{p_n^{(k)}};\ldots;\underbrace{1,\ldots,1}_{p_1^{(1)}},\ldots,\underbrace{k,\ldots,k}_{p_1^{(k)}})$$

and the total charge is $\sum_{i=1}^n \sum_{s=1}^k s p_i^{(s)}$). The coefficient of $y_1^{r_1} \cdots y_n^{r_n}$ in the second expression gives the q-character of the weight subspace $W_{\Lambda+\sum_{i=1}^n r_i\alpha_i}(\hat{\Lambda})$.

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Appendix

Table 1

Color-type	Energy	Basis
(1;2)	3	$(1_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
	4	$(1_{\alpha_2} - 4_{\alpha_1} - 1_{\alpha_1}), (0_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
	5	$(1_{\alpha_2} - 5_{\alpha_1} - 1_{\alpha_1}), (1_{\alpha_2} - 4_{\alpha_1} - 2_{\alpha_1}), (0_{\alpha_2} - 4_{\alpha_1} - 1_{\alpha_1}),$ $(-1_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
(2;2)	4	$(-1_{\alpha_2} 1_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
	5	$(-2_{\alpha_2} 1_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1}), (-1_{\alpha_2} 1_{\alpha_2} - 4_{\alpha_1} - 1_{\alpha_1})$
	6	$(-3_{\alpha_2} 1_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1}), (-2_{\alpha_2} 1_{\alpha_2} - 4_{\alpha_1} - 1_{\alpha_1}), (-2_{\alpha_2} 0_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1}),$ $(-1_{\alpha_2} 1_{\alpha_2} - 5_{\alpha_1} - 1_{\alpha_1}), (-1_{\alpha_2} 1_{\alpha_2} - 4_{\alpha_1} - 2_{\alpha_1})$

Table 2

Color-type	Energy	Color-charge-type	Basis
(1;2)	2	(1;2)	$(0_{\alpha_2} - 2_{2\alpha_1})$
	3	(1;1,1)	$(1_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
		(1;2)	$(0_{\alpha_2} - 3_{2\alpha_1}), (-1_{\alpha_2} - 2_{2\alpha_1})$
	4	(1;1,1)	$(1_{\alpha_2} - 4_{\alpha_1} - 1_{\alpha_1}), (0_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
		(1;2)	$(0_{\alpha_2} - 4_{2\alpha_1}), (-1_{\alpha_2} - 3_{2\alpha_1}), (-2_{\alpha_2} - 2_{2\alpha_1})$
	5	(1;1,1)	$(1_{\alpha_2} - 4_{\alpha_1} - 2_{\alpha_1}), (1_{\alpha_2} - 5_{\alpha_1} - 1_{\alpha_1}),$ $(0_{\alpha_2} - 4_{\alpha_1} - 1_{\alpha_1}), (-1_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
		(1;2)	$(0_{\alpha_2} - 5_{2\alpha_1}), (-1_{\alpha_2} - 4_{2\alpha_1}), (-2_{\alpha_2} - 3_{2\alpha_1}),$ $(-3_{\alpha_2} - 2_{2\alpha_1})$
(2;2)	2	(2;2)	$(0_{2\alpha_2} - 2_{2\alpha_1})$
	3	(2;2)	$(0_{2\alpha_2} - 3_{2\alpha_1}), (-1_{2\alpha_2} - 2_{2\alpha_1})$
	4	(1,1;1,1)	$(-1_{\alpha_2} 1_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
		(2;1,1)	$(0_{2\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
		(1,1;2)	$(-2_{\alpha_2} 0_{\alpha_2} - 2_{2\alpha_1})$
		(2;2)	$(0_{2\alpha_2} - 4_{2\alpha_1}), (-1_{2\alpha_2} - 3_{2\alpha_1}), (-2_{2\alpha_2} - 2_{2\alpha_1})$
	5	(1,1;1,1)	$(-1_{\alpha_2} 1_{\alpha_2} - 4_{\alpha_1} - 1_{\alpha_1}), (-2_{\alpha_2} 1_{\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
		(2;1,1)	$(0_{2\alpha_2} - 4_{\alpha_1} - 1_{\alpha_1}), (-1_{2\alpha_2} - 3_{\alpha_1} - 1_{\alpha_1})$
		(1,1;2)	$(-2_{\alpha_2} 0_{\alpha_2} - 3_{2\alpha_1}), (-3_{\alpha_2} 0_{\alpha_2} - 2_{2\alpha_1})$
		(2;2)	$(0_{2\alpha_2} - 5_{2\alpha_1}), (-1_{2\alpha_2} - 4_{2\alpha_1}), (-2_{2\alpha_2} - 3_{2\alpha_1}),$ $(-3_{2\alpha_2} - 2_{2\alpha_1})$

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